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# Coupling of Gravity Waves to Surface Currents and Ice Motion

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by

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## ABSTRACT

Perrie, W. and M. Rahman. 1991. Coupling of gravity waves to surface currents and ice motion. Can. Tech. Rep. Hydrogr. Ocean Sci. 134: 69p.

An understanding of the dynamics of ice floes in ocean waves is of considerable importance because of the development of oil and gas fields in continental shelf regions in areas where ice occurs frequently. The forces exerted by surface currents, surface gravity waves and wind fields at the ocean surface, as determined by interactions between ice, currents, waves and wind, are important in determining ice floe motion and wave conditions. This report presents mathematical models used in predicting drift motions relevant to floating ice due to wave motion and in modelling the coupling between wind, surface currents and surface waves.

## RÉSUMÉ

Perrie, W. and M. Rahman. 1991. Coupling of gravity waves to surface currents and ice motion. Can. Tech. Rep. Hydrogr. Ocean Sci. 134: 69p.

La compréhension de la dynamique des banquises dans les vagues océaniques a une importance considérable en raison de la mise en valeur des champs de pétrole et de gaz du plateau continental dans des zones où de la glace est souvent présente. Les forces exercées par les courants de surface, les vagues de gravité de surface et les champs de vent à la surface de l'océan, déterminées par les interactions de la glace, des courants, des vagues et du vent, sont importantes dans la détermination du déplacement de la banquise et des conditions des vagues. Le présent rapport montre des modèles mathématiques servant à prévoir les mouvements de dérive de la glace flottante dus au mouvement des vagues et à représenter le couplage entre le vent, les courants de surface et les vagues de surface.

# 1 INTRODUCTION

When an ice floe is forced by ocean waves, it experiences six rigid body motions: 3 translational which are surge, sway and heave, and 3 rotational, namely, roll, pitch and yaw. These motions are controlled by the incident wave field, the surface currents, the surface wind fields and the geometry and material properties of the ice. In turn, ice floes influence wave growth due to energy input to the waves by surface winds and wave dissipation due to wave breaking white-cap formation and wave-induced surface currents. Theoretical models and laboratory experiments where floes are assumed to be rigid have provided insight into the problem. However, the effects of coupled flexural response on floe motions, wave-wave energy transfer interactions and wave-ice energy transfer interactions, to name a few of the unknowns, remain unresolved.

The recent development of sophisticated motion sensors, as in the Labrador Ice Margin Experiments (LIMEX) of 1987 and 1989, to directly measure the six degrees of freedom of a given floe has resulted in a significant archive of data which should help our understanding of these processes. Floe motion sensors directly observe floe motion due to the incident wave field. Wave buoys placed in open water in the neighbourhood of the ice floe distribution provide information about the incident waves. Given assumptions concerning the dispersion relation and the transfer functions relevant for the data, we can infer relationships between the floe motion and the local wind, wave and surface current fields.

# 2 FLOE MOTION DUE TO WAVE FORCING

We assume rigid ice floes and we use the techniques of naval architecture to describe the motions resulting from forcing by ocean waves, following Squire (1981) and others. Squire (1981) found that if the ice floe dimension is small compared to the wavelength of the incident surface wave, ($\approx 1/2$ the incident wavelength), the transfer function is about unity. This is important for wave processes within the marginal ice zone (MIZ) because most floes of this region are small relative to the wavelengths of incident waves. Large floes at the ice edge in the MIZ, where short period high frequency waves penetrate, are quickly broken by wave action. Deep within the ice pack, large floes may exist because only low frequency swell waves are found: high frequency waves are quickly attenuated by ice floes in the MIZ. In practice ice floes are not as rigid as ships. Essentially, floes are elastic plates obeying Young's modulus $E \sim 6 \times 10^9 Pa$ and therefore have a flexural response coupled to rigid body motion. These bending effects, both the floes's response to the wave and the waves response to the floe, can be important.

An understanding of reflection and transmission of wave energy across the boundaries separating open water and ice covered regions in terms of wave-ice interactions has not been achieved as yet. Wadhams (1986) showed that to a first approximation, wave energy decay in ice covered water is exponential with the attenuation coefficient a function of the number of reflection coefficients across the boundaries. More recently, Liu and Mollo-Christensen (1989) suggest a wave decay that differs from exponential due to nonlinear effects. Theoretically the reflection of wave energy is a difficult problem because the velocity potentials in the two regions cannot be matched at all depths. Wadhams (1986) matched the potentials at

the surface only, recognizing that most of the wave energy is contained in the near surface region.

The drift motion on an ice floe depends on forces due to wave drift, current drag, wind drag and the Coriolis acceleration. Several models have been developed to predict ice floe drift trajectories in the open ocean. Chandler (1986) formulated a statistical model involving simple kinematics expressing ice floe velocity in terms of prevailing wind and surface currents. Mountain (1980), Sodhi and El-Tahan (1980) and Smith and Banke (1982), Ikeda (1985) developed dynamic models predicting the drift trajectories from environmental forces such as wind, currents, the Coriolis effect and sea surface slope. Hsiung and Abul-Azm (1982) developed a drift model including wave forcing. They showed that for a specific floe, even 'small' waves can result in forcing that may exceed other environmental forces on the ice floe and they suggested that inclusion of wave drift forces improved their ability to predict floe trajectories.

## 2.1 MATHEMATICAL FORMULATION

We consider a regular small amplitude wave train of height $H$, which has amplitude $A$, and angular frequency $\sigma$, propagating in water of constant depth $h$, past a floe, as depicted in

Figure 1.

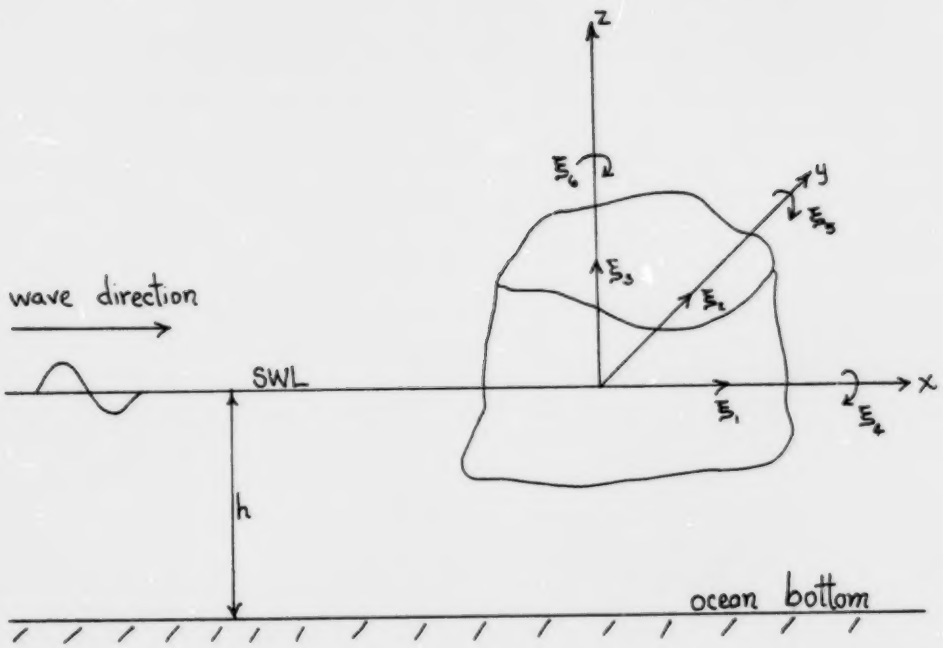



FIGURE 1. Schematic Diagram of an Ice Floe in Waves

Let $(x, y, z)$ form a Cartesian coordinate system with $x$ measured in the direction of the wave, and $z$ vertically upwards from the still water level (SWL). Consider the cylindrical coordinate system $(r, \theta, z)$ with $r$ measured radially from $z$ axis, and $\theta$ from the positive $x$ axis. Assuming the fluid is incompressible and the flow irrotational, the problem reduces to the determination of a velocity potential $\Phi$, which satisfies Laplace's equation,

$$\nabla^2 \Phi = 0 \tag{1}$$

such that the fluid velocity vector $\vec{V}$, can be defined as

$$\vec{V} = \vec{\nabla} \Phi. \tag{2}$$

The Bernoulli equation can be written as

$$p + \rho g z + \rho \frac{\partial \Phi}{\partial t} + \frac{\rho}{2}(\nabla \Phi)^2 = 0. \tag{3}$$

From the small amplitude wave theory, the surface elevation can be derived from (3) as

$$\eta = -\frac{1}{g}(\frac{\partial \Phi}{\partial t})_{z=0}. \tag{4}$$

For linear theory, the velocity potential $\Phi$, satisfies the following boundary conditions:

**Bottom Boundary Condition:**

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = -h \tag{5}$$

**Body Surface Boundary Conditions:**

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on the fixed body} \tag{6}$$

$$\frac{\partial \Phi}{\partial n} = V_n \quad \text{on the floating body} \tag{7}$$

where $n$ denotes distance in the direction of the unit vector $\vec{n}$ directed normally out from the floe's surface and $V_n$ is the velocity of the floe in the direction $n$.

In addition to conditions (5) – (7), $\Phi$ also satisfies the free surface and far field conditions. The body oscillates in the six degrees of freedom corresponding to surge, sway, heave, roll, pitch and yaw modes of oscillations as indicated in Figure 1. These are denoted by the subscripts, 1, 2, 3, 4, 5, and 6 respectively. Each mode of motion is harmonic of angular frequency, $\sigma$, and may be expressed in the complex form

$$\alpha_\ell = \xi_\ell e^{i\sigma t} \tag{8}$$

where $t$ is the time, and $\xi_\ell$ is the complex amplitude for the mode $\ell$.

In linearizing the problem, the velocity potential can be obtained by the superposition of 8 components:

$$\begin{aligned} \Phi &= Re[\phi\, e^{i\sigma t}] \\ &= Re[(\phi_0 + \phi_7 + \sum_{\ell=1}^{6} \xi_\ell \phi_\ell) e^{i\sigma t}] \end{aligned} \tag{9}$$

where

$\phi_0$ is the incident wave potential

$\phi_7$ is the scattered wave potential

$\phi_\ell$ is the forced (radiated) wave potential

and $Re$ stands for the real part. The body surface boundary condition may be applied at the equilibrium body surface $S_0$, and is given by

$$\begin{aligned} \frac{\partial}{\partial n}[Re(\phi\, e^{i\sigma t})] &= V_n \quad \text{at } S_0 \\ &= Re(v_n\, e^{i\sigma t}) \end{aligned} \tag{10}$$

where $v_n$ is the complex normal velocity. Here $V_n$ is assumed to be made up of surge, sway, heave, roll, pitch and yaw components and is given by

$$V_n = Re[\sum_{\ell=1}^{6} \frac{d\alpha_\ell}{dt} n_\ell] = Re[\sum_{\ell=1}^{6} (i\sigma) \xi_\ell n_\ell\, e^{i\sigma t}] \tag{11}$$

in which

$$\begin{aligned} n_1 &= n_x \\ n_2 &= n_y \\ n_3 &= n_z \\ n_4 &= yn_z - zn_y \\ n_5 &= zn_x - xn_z \\ n_6 &= xn_y - yn_x. \end{aligned} \tag{12}$$

where $n_x$, $n_y$ and $n_z$ are the direction cosines of $\vec{n}$ with respect to the $x$, $y$ and $z$ directions.

Equation (10) can be rewritten using (11),

$$\frac{\partial}{\partial n}[(\phi_0 + \phi_7) + \sum_{\ell=1}^{6} \phi_\ell \xi_\ell] = \sum_{\ell=1}^{6} (i\sigma)\xi_\ell n_\ell \tag{13}$$

which can be decomposed as

$$\frac{\partial}{\partial n}(\phi_0 + \phi_7) = 0 \tag{14}$$

and

$$\sum_{\ell=1}^{6} \frac{\partial \phi_\ell}{\partial n} \xi_\ell = \sum_{\ell=1}^{6} (i\sigma)\xi_\ell n_\ell \tag{15}$$

or,

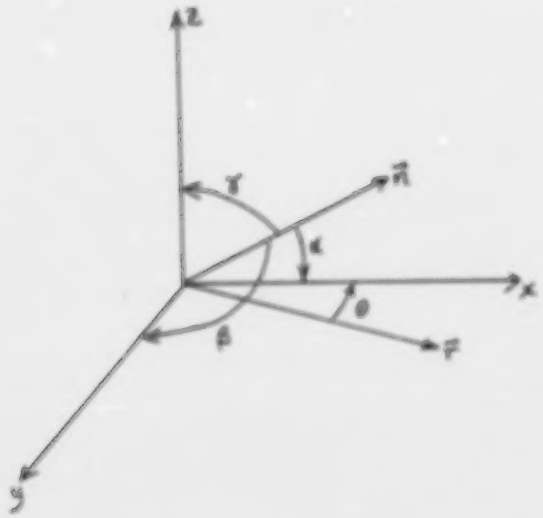
$$\frac{\partial \phi_\ell}{\partial n} = (i\sigma)n_\ell \quad \ell = 1,2,3,4,5,6. \tag{16}$$

If the normal unit vector $\vec{n}$, makes angles $\alpha, \beta, \gamma$ with $x, y, z$ axes respectively, then the direction cosines are given by (see Figure 2)

$$\begin{aligned} n_x &= \cos\alpha \\ n_y &= \cos\beta \\ n_z &= \cos\gamma \end{aligned}$$

such that

$$
\begin{aligned}
n_1 &= \cos\alpha \\
n_2 &= \cos\beta \\
n_3 &= \cos\gamma \\
n_4 &= yn_z - zn_y = (r\sin\theta)\cos\gamma - z\cos\beta \\
n_5 &= zn_x - xn_z = z\cos\alpha - (r\cos\theta)\cos\gamma \\
n_6 &= xn_y - yn_x = (r\cos\theta)\cos\beta - (r\sin\theta)\cos\alpha
\end{aligned}
\tag{17}
$$




$x = r\cos\theta$

$y = r\sin\theta$

FIGURE 2: Sketch of Direction Cosines

The body surface boundary conditions (14) and (16) are then reduced to

$$
\frac{\partial\phi_T}{\partial n} = -\frac{\partial\phi_0}{\partial n} \tag{18}
$$

$$\begin{cases} \frac{\partial \phi_1}{\partial n} = (i\sigma)\cos\alpha & \text{for } \ell = 1 \\ \frac{\partial \phi_2}{\partial n} = (i\sigma)\cos\beta & \text{for } \ell = 2 \\ \frac{\partial \phi_3}{\partial n} = (i\sigma)\cos\gamma & \text{for } \ell = 3 \\ \frac{\partial \phi_4}{\partial n} = (i\sigma)(r\sin\theta\cos\gamma - z\cos\beta) & \text{for } \ell = 4 \\ \frac{\partial \phi_5}{\partial n} = (i\sigma)(z\cos\alpha - r\cos\theta\cos\gamma) & \text{for } \ell = 5 \\ \frac{\partial \phi_6}{\partial n} = (i\sigma)(r\cos\theta\cos\beta - r\cos\theta\cos\alpha) & \text{for } \ell = 6 \end{cases} \tag{19}$$

The incident wave potential $\phi_0$ in finite water depth may be written as

$$\phi_0 = (-\frac{gA}{\sigma})\frac{\cosh k(z+h)}{\cosh kh}e^{-ikx} \tag{20}$$

which corresponds to the incident wave of amplitude $A$

$$\eta_0 = A\sin(kx - \sigma t) \tag{21}$$

Equation (20) can be expressed in terms of Bessel Series as follows:

$$\phi_0 = (-\frac{gA}{\sigma})\frac{\cosh k(z+h)}{\cosh kh}\sum_{n=0}^{\infty}\beta_n J_n(kr)\cos n\theta \tag{22}$$

in which $g$ is the gravitational constant, $k$ is the wave number, $J_n$ is the Bessel function of first kind, of order $n$; $\beta_0 = 1$ and $\beta_n = 2(-i)^n$ for $n \geq 1$.

Equation (22) can also be written as

$$\phi_0 = (-\frac{gA}{\sigma})\frac{\cosh k(z+h)}{\cosh kh}\sum_{n=-\infty}^{\infty}(-i)^n J_n(kr)e^{in\theta} \tag{23}$$

It is necessary to emphasize here that the potential, $\phi_\ell$, $\ell = 1, 2, 3, 4, 5, 6$ and 7 must satisfy the Laplace equation

$$\nabla^2\phi_\ell = 0. \tag{24}$$

Bottom boundary conditions are

$$\frac{\partial \phi_\ell}{\partial z} = 0 \quad \text{at } z = -h \tag{25}$$

The linearized Kinematic and Dynamic boundary conditions at the free surface require

$$\frac{\partial \phi_\ell}{\partial z} - \frac{\sigma^2}{g}\phi_\ell = 0 \quad \text{at } z = 0 \tag{26}$$

The radiation condition, ensuring that the scattered potentials correspond to outgoing waves, is

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \phi_\ell}{\partial r} - ik\phi_\ell\right) = 0 \quad \text{for } \ell = 1,2,3,4,5,6 \text{ and } 7 \tag{27}$$

together with the surface boundary conditions given by (19).

The usual way of determining the solution of this boundary value problem is to represent the unknown potentials, $\phi_\ell$, $\ell = 1, 2, \cdots 7$ by a distribution of sources over the equilibrium body, $S_0$. Thus, $\phi_\ell(\vec{x})$ at a general field point $\vec{x} = (x, y, z)$ is expressed as (e.g. Wehausen and Laitone (1960))

$$\phi_\ell(\vec{x}) = \frac{1}{4\pi}\int\int_{S_0} f_\ell(\vec{X})G(\vec{x},\vec{X})dS \quad \text{for } \ell = 1,2,3,\cdots 7. \tag{28}$$

in which $f_\ell(\vec{X})$ is a source distribution function at $\vec{X} = (X, Y, Z,)$, a point on $S_0$, and $G(\vec{x},\vec{X})$ a Green's function for the general point $\vec{x}$ due to a source of unit strength at $\vec{X}$. The Green's function must satisfy the Laplace equation, together with the bottom boundary condition, a free surface boundary condition and the radiation boundary condition. Thus, $\phi_\ell$ will also satisfy these equations and it remains to be chosen so as to ensure that $f_\ell$ satisfies the body surface boundary conditions (14) and (16).

The appropriate Green's function, first derived by John (1950) for finite depth, can be

expressed as

$$\begin{aligned} G(\vec{x},\vec{X}) &= \frac{1}{R}+\frac{1}{R'} \\ &\quad +2(P.V.)\int_0^\infty \frac{(\mu+\nu)e^{-\mu h}\cosh(\mu(Z+h))\cosh(\mu(z+h))}{\mu\sinh(\mu h)-\nu\cosh(\mu h)}J_0(\mu r)d\mu \\ &\quad -i\,C_0\cosh(k(Z+h))\cosh(k(z+h))J_0(kr) \end{aligned} \tag{29}$$

where

$$\begin{aligned} R &= [(x-X)^2+(y-Y)^2+(z-Z)^2]^{\frac{1}{2}} \\ R' &= [(x-X)^2+(y-Y)^2+(z+2h+Z)^2]^{\frac{1}{2}} \\ r &= \sqrt{(x-X)^2+(y-Y)^2} \\ k\tanh(kh) &= \frac{\sigma^2}{g}=\nu \\ C_0 &= \frac{2\pi(\nu^2-k^2)}{(k^2-\nu^2)h+\nu} \end{aligned} \tag{30}$$

where *P.V.* stands for the Cauchy-Principal value of the integral.

The second equivalent form of the Green's function involves a series representation

$$\begin{aligned} G(\vec{x},\vec{X}) &= i\,C_0\cosh(k(Z+h))\cosh(k(z+h))H_0^{(2)}(kr) \\ &\quad +4\sum_{m=1}^{\infty} C_m\cos(\mu_m(z+h))\cos(\mu_m(Z+h))K_0(\mu_m r) \end{aligned} \tag{31}$$

where $K_0$ denotes the modified Bessel function of the second kind and order zero, and $H_0^{(2)}$ the Hankel function of order zero.

Here,

$$C_m = \frac{\nu^2+\mu_m^2}{(\mu_m^2+\nu^2)h-\nu}$$

and $\mu_m$ are the real positive roots of the equation

$$\mu_m\tan(\mu_m h)+\nu=0 \tag{32}$$

taken in ascending order. When $(kr) > 0.1$, the series form of Green's function (32) converges sufficiently rapidly, whereas when $(kr) < 0.1$, the integral form (29) is more convenient (Garrison and Chow (1972)).

The body surface boundary conditions (14) and (16) give rise to a set of seven surface integrals for the unknown source density function $f_\ell$ (see Kellogg (1953)).

$$-\frac{1}{2}f_\ell(\vec{x}) + \frac{1}{4\pi}\int\int_{S_0} f_\ell(\vec{X})\frac{\partial G}{\partial n}(\vec{x},\vec{X})dS = g_\ell(\vec{x}) \quad \ell = 1,2,3,\cdots 7 \tag{33}$$

where

$$\left.\begin{aligned} g_1(\vec{x}) &= (i\sigma)\cos\alpha \\ g_2(\vec{x}) &= (i\sigma)\cos\beta \\ g_3(\vec{x}) &= (i\sigma)\cos\gamma \\ g_4(\vec{x}) &= (i\sigma)(y\cos\gamma - z\cos\beta) \\ g_5(\vec{x}) &= (i\sigma)(z\cos\alpha - x\cos\gamma) \\ g_6(\vec{x}) &= (i\sigma)(x\cos\beta - y\cos\alpha) \\ g_7(\vec{x}) &= -\frac{\partial\phi_0}{\partial n} \end{aligned}\right\} \tag{34}$$

in which

$$\begin{aligned} \frac{\partial\phi_0}{\partial n} &= \frac{\partial\phi_0}{\partial x}n_x + \frac{\partial\phi_0}{\partial y}n_y + \frac{\partial\phi_0}{\partial z}n_z \\ &= -\frac{gAk}{\sigma}e^{-ikx}[-in_x\frac{\cosh k(z+h)}{\cosh kh} + n_z\frac{\sinh k(z+h)}{\cosh kh}] \end{aligned}$$

Introducing the values for $n_x$ and $n_z$, we obtain

$$\begin{aligned} g_7(\vec{x}) &= -\frac{\partial\phi_0}{\partial n} \\ &= (\frac{gAk}{\sigma})e^{-ikx}[-i\cos\alpha\frac{\cosh k(z+h)}{\cosh kh} + \cos\gamma\frac{\sinh k(z+h)}{\cosh kh}] \end{aligned} \tag{35}$$

Equation (33) is a Fredholm integral equation of second kind which must be solved for $f_\ell(\vec{x})$. This can be done by discretizing the whole body into a finite number $N$ of small segments

with $f_\ell(\vec{X})$ assumed uniform over each segment. Hence, equation (33) leads to a set of $N$ matrix equations that can be solved numerically to give $f_\ell(\vec{X})$ for each segment. It is to be noted that each segment contains seven integral equations. Once $f_\ell(\vec{X})$ is known, the potential $\phi_\ell(\vec{x})$ can be determined from the integral equation (28).

The equation of motion of the floating body may be written as

$$m_{i\ell}\frac{d^2\alpha_\ell}{dt^2} = -C_{i\ell}\alpha_\ell + \rho\int\int_{S_0}\frac{\partial}{\partial t}(\phi\, e^{i\sigma t})n_i dS \tag{36}$$

where the first term is an inertia force, second term is a restoring force and the third term contains the radiated force and the exciting force. The third term can be decomposed as follows:

$$\begin{aligned} F_i &= \rho\int\int_{S_0}\frac{\partial}{\partial t}(\phi\, e^{i\sigma t})n_i dS \\ &= (F_i^{(e)} + F_i^{(f)})e^{i\sigma t} \end{aligned}$$

where

$$F_i^{(e)} = (i\sigma\rho)\int\int_{S_0}(\phi_0 + \phi_7)n_i dS \tag{37}$$

$$F_i^{(f)} = (i\sigma\rho)\sum_{\ell=1}^{6}\int\int_{S_0}\phi_\ell\xi_\ell n_i dS \tag{38}$$

and $F_i^{(e)}$ and $F_i^{(f)}$ are defined to be exciting and radiated forces respectively. We further decompose the radiated force $F_i^{(f)}$ in the following manner:

$$F_i^{(f)}e^{i\sigma t} = \sum_{\ell=1}^{6}\{\mu_{i\ell}\frac{\partial^2\alpha_\ell}{\partial t^2} + \lambda_{i\ell}\frac{\partial\alpha_\ell}{\partial t}\} \tag{39}$$

where $\mu_{i\ell}$ is called <u>added</u> mass coefficient and $\lambda_{i\ell}$ is the <u>damping</u> coefficient. (Note that $\mu_{i\ell}$ and $\lambda_{i\ell}$ are both real). Thus Equation (36) can be written as

$$(m_{i\ell} + \mu_{i\ell})\frac{\partial^2\alpha_\ell}{\partial t^2} + \lambda_{i\ell}\frac{\partial\alpha_\ell}{\partial t} + C_{i\ell}\alpha_\ell = F_i^{(e)}e^{i\sigma t} \tag{40}$$

which subsequently reduces to

$$\sum_{\ell=1}^{6}\{-\sigma^2(m_{i\ell}+\mu_{i\ell})+i\sigma\lambda_{i\ell}+C_{i\ell}\}\xi_\ell = F_i^{(e)} \quad \text{for } i=1,2,\cdots 6. \tag{41}$$

Equating (38) and (39), we obtain

$$(i\sigma\rho)\int\int_{S_0}\phi_\ell n_i dS = -\sigma^2\mu_{i\ell}+(i\sigma)\lambda_{i\ell} \tag{42}$$

which clearly implies that

$$\left.\begin{aligned}\mu_{i\ell} &= \tfrac{\rho}{\sigma} I_m\{\textstyle\int\int_{S_0}\phi_\ell n_i dS\}\\ \lambda_{i\ell} &= \rho Re\{\textstyle\int\int_{S_0}\phi_\ell n_i dS\}\end{aligned}\right\} \tag{43}$$

$$\ell = 1,2,3,\cdots 6$$

$$i = 1,2,3,\cdots 6.$$

Once $F_i^{(e)}$, $\mu_{i\ell}$ and $\lambda_{i\ell}$ are known, it is a relatively easy matter to solve the equations of motion of the floe given by Equation (41). Having obtained the motion, the velocity potential (9) can be determined. Once we know the potential, the pressure at any point of the body can be determined and subsequently the dynamic load can be obtained. Furthermore, we can find the free surface elevation at any point and subsequently the scattering amplitude. The fluid velocity and acceleration can also be readily determined.

## 2.2 SECOND ORDER WAVE DRIFT FORCES

In the drift motion calculation to be described in Section 2.4, wave drift forces are important. This may be shown from diffraction theory. In the calculation of drift motion, the floe is assumed to have two modes of oscillations, namely surge and sway. The corresponding drift

force components are therefore required. These wave drift forces may be obtained by a far field momentum formulation, or direct integration of the fluid pressure on the submerged surface of the floating body. We shall follow the latter approach due to Lighthill (1979) who demonstrates that the second order force on the body surface is the sum of a waterline force $F_w$ and a dynamic force $F_d$. It is to be noted that these forces are due to the first order wave potential, and the second order potential is not required.

We know the pressure can be obtained from the Bernoulli equation

$$P = -\rho\frac{\partial\Phi}{\partial t} - \rho g z - \frac{1}{2}\rho(\nabla\Phi)^2. \tag{44}$$

The first order horizontal force components are given by

$$\vec{F}^{(1)} = \int\int_{S_0} \rho(\frac{\partial\Phi}{\partial t})\vec{n}dS \tag{45}$$

whereas the second order horizontal forces can be obtained from

$$\begin{aligned} \vec{F}^{(2)} &= \int\int_{S_0}(\rho g z)\vec{n}dS + \int\int_{S_0}\frac{\rho}{2}(\nabla\Phi)^2\vec{n}dS \\ &= \vec{F}_w^{(2)} + \vec{F}_d^{(2)} \end{aligned} \tag{46}$$

where

$$\begin{aligned} \vec{F}_w^{(2)} &= \text{waterline force} \\ &= \int\int_{S_0}(\rho g z)\vec{n}dS \\ &= \oint_w\int_0^\eta(\rho g z)\vec{n}dz\,ds \\ &= \frac{\rho g}{2}\oint_w \eta^2\vec{n}ds \\ &= \frac{\rho}{2g}\oint_w(\frac{\partial\Phi}{\partial t})^2_{z=0}\vec{n}ds \end{aligned} \tag{47}$$

and

$$\vec{F}_d^{(2)} = \text{dynamic force}$$
$$= \int\int_{S_0} \frac{\rho}{2}(\nabla\Phi)^2 \vec{n} dS \tag{48}$$

For the case of a circular cylindrical floe of radius $a$,

$$\begin{Bmatrix} F_{wx}^{(2)} \\ F_{wy}^{(2)} \end{Bmatrix} = \frac{\rho}{2g}\int_0^{2\pi} (\frac{\partial\Phi}{\partial t})^2_{\substack{z=0 \\ r=a}} \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} d\theta \tag{49}$$

$$\begin{Bmatrix} F_{dx}^{(2)} \\ F_{dy}^{(2)} \end{Bmatrix} = \frac{\rho}{2}\int_{-h}^{0}\int_0^{2\pi} (\nabla\Phi)^2 \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} d\theta dz \tag{50}$$

Equations (49) and (50) both contain the oscillatory and steady-state part. In a drift force calculation, we shall be mainly concerned with only the steady-state part. From equation (9), we write

$$(\frac{\partial\Phi}{\partial t})^2 = \frac{1}{2}Re[-\sigma^2\phi^2 e^{2i\sigma t} + \sigma^2\phi\phi^*] \tag{51}$$

and

$$(\nabla\Phi)^2 = \frac{1}{2}Re[(\nabla\phi)^2 e^{2i\sigma t} + (\nabla\phi)(\nabla\phi)^*] \tag{52}$$

Thus, (49) and (50) can be written as

$$\begin{Bmatrix} F_{wx}^{(2)} \\ F_{wy}^{(2)} \end{Bmatrix} = \frac{\rho a\sigma^2}{4g} Re\{\int_0^{2\pi} (-\phi^2 e^{2i\sigma t} + \phi\phi^*) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} d\theta\} \tag{53}$$

$$\begin{Bmatrix} f_{dx}^{(2)} \\ F_{dy}^{(2)} \end{Bmatrix} = \frac{\rho a}{4} Re\{\int_{-h}^{0}\int_0^{2\pi} \{(\nabla\phi)^2 e^{2i\sigma t} + (\nabla\phi)(\nabla\phi)^*\}\} \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} d\theta\, dz \tag{54}$$

The first integrals on the right of (53) and (54) are oscillatory, whereas the second integrals are time-independent. The second integrals are defined as the drift forces and can be written

as

$$\begin{Bmatrix} < F_{wx}^{(2)} > \\ < F_{wy}^{(2)} > \end{Bmatrix} = (\frac{\rho a \sigma^2}{4g}) \int_0^{2\pi} \phi \phi^* \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} d\theta \tag{55}$$

$$\begin{Bmatrix} < F_{dx}^{(2)} > \\ < F_{dy}^{(2)} > \end{Bmatrix} = \frac{\rho a}{4} \int_{-h}^{0} \int_0^{2\pi} (\nabla\phi)(\nabla\phi)^* \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} d\theta \, dz \tag{56}$$

where $\phi$ and $\nabla\phi$ are evaluated at $r = a$. In addition to the wave drift forces, the zero frequency added masses of the floe body with respect to the surge and sway motions are also required in the drift motion calculations. These can be obtained from wave diffraction calculations using very low wave frequency. The Green's function may be modified to obtain the zero frequency added mass, following Garrison (1978).

## 2.3 DRIFT MOTIONS

With respect to drift motions, a floe essentially possesses three degrees of freedom: surge, sway and yaw modes of oscillations. In practice, the rotational component yaw, about $z$ axis, is very small and can be neglected. Forces acting on the floe generally include wave drift, current drag, wind drag and the Coriolis effect. The equations for the drift motion are then

$$(m + \mu_x)\frac{du}{dt} = F_x^{(d)} + F_x^{(w)} + F_x^{(c)} + F_x^{(r)} \tag{57}$$

$$(m + \mu_y)\frac{dv}{dt} = F_y^{(d)} + F_y^{(w)} + F_y^{(c)} + F_y^{(r)} \tag{58}$$

where

$m$ = mass of the floating structure

$\mu_x, \mu_y$ = the zero frequency added masses in the $x$ and $y$ directions

$F^{(d)}_{x,y}$ = the horizontal drift forces in the $x$ and $y$ directions

$F^{(w)}_{x,y}$ = the horizontal wind drag forces in the $x$ and $y$ directions

$F^{(c)}_{x,y}$ = the horizontal current drag forces in the $x$ and $y$ directions

$F^{(r)}_{x,y}$ = the horizontal Coriolis forces in the $x$ and $y$ directions

and the form of the drift forces $F^{(d)}_{x,y}$ are given in equations (55) and (56). The current drag forces are given by

$$F^{(c)}_x = \frac{1}{2}\rho A C_D (u^{(c)} - u)\sqrt{(u^{(c)} - u)^2 + (v^{(c)} - v)^2} \tag{59}$$

$$F^{(c)}_y = \frac{1}{2}\rho A C_D (v^{(c)} - v)\sqrt{(v^{(c)} - v)^2 + (u^{(c)} - u)^2} \tag{60}$$

where $\vec{u}^{(c)}$ is the local current velocity vector, and $\vec{u}$ the drift velocity of the floe. Wind stresses can be written as

$$F^{(w)}_x = \frac{C_1}{(\eta + h)^2}|\vec{W}|w_x \tag{61}$$

$$F^{(w)}_y = \frac{C_1}{(\eta + h)^2}|\vec{W}|w_y \tag{62}$$

where

$|\vec{W}| = \sqrt{w_x^2 + w_y^2}$

$w_x, w_y$ = are the wind velocity components $(u^{(w)} - u)$ and $(v^{(w)} - v)$

$C_1 = 0.35 \times 10^{-6} kg/m^2$

$\eta$ = wave elevation

$h$ = water depth

The Coriolis force can be written as

$$F_x^{(r)} = +\rho\Omega v \tag{63}$$

$$F_y^{(r)} = -\rho\Omega u \tag{64}$$

where $\rho$ is the water density, $\Omega = 2w\sin\phi$, with $w$ the angular velocity of the earth around its axis (rad/s) and $\phi$, the latitude. The water elevation $\eta$, which must be evaluated from equations (4) and (9)

$$\begin{aligned}\eta &= -\frac{1}{g}(\frac{\partial\Phi}{\partial t})_{z=0} \\ &= -\frac{\sigma}{g}Re[(\phi_0 + \phi_7 + \sum_{\ell=1}^{6}\phi_\ell\xi_\ell)i\, e^{i\sigma t}]\end{aligned} \tag{65}$$

## Special Cases

Isaacson (1986) showed that the wave drift force and the current drag force may be more significant than the wind drag force and the Coriolis force in determining the drift motion of a floe. If the local current $u^{(c)}$ is uniform, then the drift equations of motion (57) and (58) may be written as

$$\begin{aligned}(m+\mu_x)\frac{du}{dt} &= F_x^{(d)} + \frac{1}{2}\rho AC_D(u^{(c)}-u)\sqrt{(u^{(c)}-u)^2+(v^{(c)}-v)^2} \\ (m+\mu_y)\frac{dv}{dt} &= F_y^{(d)} + \frac{1}{2}\rho AC_D(v^{(c)}-v)\sqrt{(u^{(c)}-u)^2+(v^{(c)}-v)^2}\end{aligned} \tag{66}$$

The transformations

$$\begin{aligned}u^{(c)} - u &= U \\ v^{(c)} - v &= V\end{aligned} \tag{67}$$

reduce equations (66) to

$$\begin{aligned}(m+\mu_x)\frac{dU}{dt} &= -F_x^{(d)} - \frac{1}{2}\rho AC_D U\sqrt{U^2+V^2} \\ (m+\mu_y)\frac{dV}{dt} &= -F_y^{(d)} - \frac{1}{2}\rho AC_D V\sqrt{U^2+V^2}\end{aligned} \tag{68}$$

**Case I:**

If there is no drag force, then $C_D = 0$ and the drift motion is given by

$$\begin{aligned} U &= -\frac{F_x^{(d)}}{m+\mu_x}t + U_0 \\ V &= -\frac{F_y^{(d)}}{m+\mu_y}t + V_0 \end{aligned} \tag{69}$$

where initially

$$\begin{aligned} U_0 &= u^{(c)} - u_0 \\ V_0 &= u^{(c)} - v_0 \end{aligned}$$

**Case II:**

If $\frac{1}{2}\rho A C_D\sqrt{U^2+V^2} = \lambda$ a constant, then equations (68) can be written as

$$\begin{aligned} (m+\mu_x)\frac{dU}{dt} + \lambda U &= -F_x^{(d)} \\ (m+\mu_y)\frac{dV}{dt} + \lambda V &= -F_y^{(d)} \end{aligned} \tag{70}$$

and solutions are

$$\begin{aligned} U &= -\frac{F_x^{(d)}}{\lambda} + (U_0 + \frac{F_x^{(d)}}{\lambda})e^{-\frac{\lambda t}{m+\mu_x}} \\ V &= -\frac{F_y^{(d)}}{\lambda} + (V_0 + \frac{F_y^{(d)}}{\lambda})e^{-\frac{\lambda t}{m+\mu_y}} \end{aligned} \tag{71}$$

**Case III:**

If floe drifts in the $x$ direction and the drift force is negligible in comparison with the current drag force, then the equation of drift motion is

$$(m+\mu)\frac{dU}{dt} + \frac{1}{2}\rho A C_D U^2 = 0, \tag{72}$$

implying that

$$U = \frac{U_0}{1 + \frac{\rho A C_D U_0 t}{2(m+\mu)}}. \tag{73}$$

and from the transformation of (67),

$$u(t) = u^{(c)} - \frac{u^{(c)} - u_0}{1 + \frac{\rho A C_D (u^{(c)} - u_0) t}{2(m+\mu)}}. \tag{74}$$

**Case IV:**

Let us assume the Coriolis and drift forces are significant. In this situation, the drift equations of motion are

$$\left.\begin{aligned} \frac{du}{dt} - k_1 v &= F_1 \\ \frac{dv}{dt} + k_2 u &= F_2 \end{aligned}\right\} \tag{75}$$

where

$$\left.\begin{aligned} k_1 = \frac{\rho Q}{m+\mu_x} \quad , \quad k_2 = \frac{\rho Q}{m+\mu_y} \\ F_1 = \frac{F_x^{(a)}}{m+\mu_x} \quad , \quad F_2 = \frac{F_y^{(a)}}{m+\mu_y} \end{aligned}\right\} \tag{76}$$

and $k_1$, $k_2$, $F_1$ and $F_2$ are constants. If the floe starts from rest at time $t = 0$, then $u(0) = 0$ and $v(0) = 0$. Laplace transforming, (75) leads to

$$\left.\begin{aligned} s^2\mathcal{L}(u) - k_1 s\mathcal{L}(v) &= F_1 \\ k_2 s\mathcal{L}(u) + s^2\mathcal{L}(v) &= F_2 \end{aligned}\right\} \tag{77}$$

from which we infer

$$\mathcal{L}(u) = \frac{sF_1 + k_1 F_2}{s(s^2 + k_1 k_2)} \tag{78}$$

$$\mathcal{L}(v) = \frac{sF_2 - k_2 F_1}{s(s^2 + k_1 k_2)} \tag{79}$$

Inverting these transforms, we obtain

$$\begin{aligned} u(t) &= \frac{F_1}{\sqrt{k_1 k_2}} \sin(\sqrt{k_1 k_2}t) + \frac{F_2}{k_2}[1 - \cos\sqrt{k_1 k_2}t)] \\ v(t) &= \frac{F_2}{\sqrt{k_1 k_2}} \sin(\sqrt{k_1 k_2}t) - \frac{F_1}{k_1}[1 - \cos\sqrt{k_1 k_2}t)] \end{aligned} \tag{80}$$

These two expressions show the oscillatory behaviour of the drift motion when the Coriolis effect is assumed.

## 2.4 NUMERICAL SOLUTION BY USING FOURTH-ORDER RUNGE-KUTTA METHOD

Equations (58) and (59) may generally be written as

$$\frac{du}{dt} = f_1(t, x, y, u, v) \tag{81}$$

$$\frac{dv}{dt} = f_2(t, x, y, u, v) \tag{82}$$

where

$$\frac{dx}{dt} = f_3(t, x, y, u, v) = u \tag{83}$$

$$\frac{dy}{dt} = f_4(t, x, y, u, v) = v \tag{84}$$

and

$$\left.\begin{aligned} f_1 &= (F_x^{(d)} + F_x^{(w)} + F_x^{(c)} + F_x^{(r)})/(m + \mu_a) \\ f_2 &= (F_y^{(d)} + F_y^{(w)} + F_y^{(c)} + F_y^{(r)})/(m + \mu_a) \end{aligned}\right\} \tag{85}$$

The fourth order Runge-Kutta numerical scheme implies the solution of the $(j+1)^{th}$ time step, as

$$\left.\begin{aligned} u^{j+1} &= u^j + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ v^{j+1} &= v^j + \tfrac{1}{6}(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4) \\ x^{j+1} &= x^j + \tfrac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \\ y^{j+1} &= y^j + \tfrac{1}{6}(n_1 + 2n_2 + 2n_3 + n_4) \end{aligned}\right\} \tag{86}$$

where

$$\left.\begin{aligned} k_1 &= (\Delta t) f_1(t, x, y, u, v) \\ \ell_1 &= (\Delta t) f_2(t, x, y, u, v) \\ m_1 &= (\Delta t) f_3(t, x, y, u, v) \\ n_1 &= (\Delta t) f_4(t, x, y, u, v) \end{aligned}\right\} \tag{87}$$

$$\left.\begin{aligned} k_2 &= (\Delta t) f_1(t + \tfrac{\Delta t}{2}, x + \tfrac{m_1}{2}, y + \tfrac{n_1}{2}, u + \tfrac{k_1}{2}, v + \tfrac{\ell_1}{2}) \\ \ell_2 &= (\Delta t) f_2(t + \tfrac{\Delta t}{2}, x + \tfrac{m_1}{2}, y + \tfrac{n_1}{2}, u + \tfrac{k_1}{2}, v + \tfrac{\ell_1}{2}) \\ m_2 &= (\Delta t) f_3(t + \tfrac{\Delta t}{2}, x + \tfrac{m_1}{2}, y + \tfrac{n_1}{2}, u + \tfrac{k_1}{2}, v + \tfrac{\ell_1}{2}) \\ n_2 &= (\Delta t) f_4(t + \tfrac{\Delta t}{2}, x + \tfrac{m_1}{2}, y + \tfrac{n_1}{2}, u + \tfrac{k_1}{2}, v + \tfrac{\ell_1}{2}) \end{aligned}\right\} \tag{88}$$

$$\left.\begin{aligned} k_3 &= (\Delta t) f_1(t + \tfrac{\Delta t}{2}, x + \tfrac{m_2}{2}, y + \tfrac{n_2}{2}, u + \tfrac{k_2}{2}, v + \tfrac{\ell_2}{2}) \\ \ell_3 &= (\Delta t) f_2(t + \tfrac{\Delta t}{2}, x + \tfrac{m_2}{2}, y + \tfrac{n_2}{2}, u + \tfrac{k_2}{2}, v + \tfrac{\ell_2}{2}) \\ m_3 &= (\Delta t) f_3(t + \tfrac{\Delta t}{2}, x + \tfrac{m_2}{2}, y + \tfrac{n_2}{2}, u + \tfrac{k_2}{2}, v + \tfrac{\ell_2}{2}) \\ n_3 &= (\Delta t) f_4(t + \tfrac{\Delta t}{2}, x + \tfrac{m_2}{2}, y + \tfrac{n_2}{2}, u + \tfrac{k_2}{2}, v + \tfrac{\ell_2}{2}) \end{aligned}\right\} \tag{89}$$

and

$$\left.\begin{aligned} k_4 &= (\Delta t) f_1(t+\Delta t, x+m_3, y+n_3, u+k_3, v+\ell_3) \\ \ell_4 &= (\Delta t) f_2(t+\Delta t, x+m_3, y+n_3, u+k_3, v+\ell_3) \\ m_4 &= (\Delta t) f_3(t+\Delta t, x+m_3, y+n_3, u+k_3, v+\ell_3) \\ n_4 &= (\Delta t) f_4(t+\Delta t, x+m_3, y+n_3, u+k_3, v+\ell_3) \end{aligned}\right\} \qquad (90)$$

Equation (86) specifies the floe velocity and location at the new time $t + \Delta t$ which may be obtained in terms of time $t$ variables.

# 3 DIFRACTION CAUSED BY WAVE-ICE INTER ACTION: A BOUNDARY ELEMENT METHOD

We assume the ice floe is a fixed rigid body subjected to fluid motion from a distant origin, neglect surface gravity waves and assume classical inviscid irrotational two-dimensional fluid flow. As a first step to a more complicated problem, we use a boundary element method. Martin (1969) used a finite element method to solve a flow problem around a cylinder between parallel plates. Alternately, Brebbia (1984) later analysed the same problem by a boundary element approach thereby demonstrating the efficiency of the method. There have been further applications of boundary element methods to fluid motion problems by Connor and Brebbia (1977) and others. However, applications of a boundary element method to physical problems of this kind are limited.

## 3.1 MATHEMATICAL FORMULATION

We assume a right-handed, rectangular Cartesian coordinate system in which the $z$ axis is vertically upwards. The $x$ and $y$ axes lie on the free surface when the fluid is at rest. We

consider the fluid to be inviscid, incompressible, and its motion is irrotational and hence there exists a velocity potential. We are concerned with a simple harmonic plane progressive wave of small amplitude obliquely incident upon a fixed/floating large ice floe. The direction of propagation of the incident wave makes an angle $\alpha$ with the $x$ axis. We propose to investigate the wave diffraction caused by the presence of the floe. The numerical procedure used is the boundary element method.

The fluid motion can be described by a linearized velocity potential $\Phi(x, y, z, t)$, which satisfies Laplace's equation within the fluid domain shown in Figure 3,

$$\nabla^2 \Phi = 0 \tag{91}$$

subject to the boundary conditions:

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0 \qquad \text{on } C_F \tag{92}$$

$$\frac{\partial \Phi}{\partial n} = 0 \qquad \text{on } C_B \tag{93}$$

$$\frac{\partial \Phi}{\partial z} = 0 \qquad \text{on } C_D \tag{94}$$

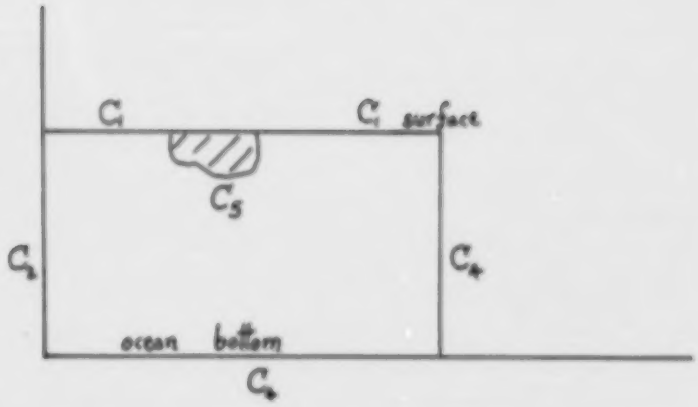



Figure 3: Definition Sketch

where $g$ is the acceleration due to gravity, $n$ denotes distance in the direction of the unit normal vector $\vec{n}$ directed outwards from the fluid region, $h$ is the depth of the ocean, and $t$ is the time. With reference to Figure 3, $C_B$ is the submerged body surface, $C_F$ is the free surface, $C_R$ is the radiation surface far away from the body and $C_D$ is the bottom boundary surface. In addition to conditions (92) and (94), we need to impose a radiation boundary condition when $r \to \infty$ to make the solution unique. This condition is introduced below.

We assume that the linearized velocity potential $\Phi(x, y, z, t)$ has a form

$$\Phi(x, y, z, t) = Re[\phi(x, z)\exp\{i(k_y y - \sigma t)\}] \tag{95}$$

where $k_y$is the wave number component in the y direction, and $\sigma$ is the frequency. Using (95) in equations (91) to (94) yields:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - k_y^2\phi = 0 \quad \text{in } R \tag{96}$$

$$\frac{\partial\phi}{\partial z} - \frac{\sigma^2}{g}\phi = 0 \quad \text{on } C_F \tag{97}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } C_B \tag{98}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on } C_D \text{ or } z = -h \tag{99}$$

Here (96) is known as the Helmholtz equation in two dimensions.

To impose the radiation condition it is convenient to assume that

$$\phi(x, z) = \phi_I(x, z) + \phi_s(x, z) \tag{100}$$

where $\phi_I$ and $\phi_S$ are, respectively, the complex potentials for the incident and diffracted waves. It is well-known that the incident wave potential $\phi_I$ of amplitude $A$ is

$$\phi_I = \frac{gA}{\sigma}\frac{\cosh k(z + h)}{\cosh kh}e^{ik_x x} \tag{101}$$

where

$$\sigma^2 = gk\tanh kh$$

$$k_x = k\cos\alpha$$

$$k_y = k\sin\alpha$$

The diffracted potential $\phi_s(x,z)$ must satisfy equations (96) to (99) and can be explicitly written as

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}-k_y^2)\phi_s(x,z) = 0 \quad \text{in } R \tag{102}$$

$$(\frac{\partial}{\partial z}-\frac{\sigma^2}{g})\phi_s = 0 \quad \text{on } C_F \tag{103}$$

$$\frac{\partial\phi_s}{\partial n} = -\frac{\partial\phi_I}{\partial n} \quad \text{on } C_B \tag{104}$$

$$\frac{\partial\phi_s}{\partial z} = 0 \quad \text{on } C_D \text{ or } z=-h \tag{105}$$

It is to be noted here that the diffracted potential $\phi_s$ should satisfy the Orr-Somerfeld radiation condition

$$(\frac{\partial}{\partial n}\pm ik_x)\phi_s \Rightarrow 0 \quad \text{as } |x|\to\infty \quad \text{on } C_R \tag{106}$$

Therefore, to satisfy this condition we require that the diffracted potential $\phi_s$ has an asymptotic behaviour as $|x|\to\infty$.

$$\phi_s \sim \frac{gA}{\sigma}\cosh k(z+h)\cosh kh \left\{ \begin{matrix} R \\ T \end{matrix} \right\} \exp(\pm i\,k_x x) \tag{107}$$

where $R$ and $T$ are respectively the reflexion and transmission coefficients. These two coefficients are obtained by Bai (1977) for the case of a rectangular block floating in the ocean. In our boundary element analysis we shall make use of his work.

## 3.2 BOUNDARY ELEMENT FORMULATION

Dropping the subscript $s$ from the diffracted potential $\phi_s(x,z)$, we can consider a boundary value problem described for complex function $\phi(x,z)$ as

$$(\nabla^2 - k_y^2)\phi(x,z) = 0 \quad \text{in } R \tag{108}$$

with two types of boundary conditions as given below:

(I) The mixed type boundary condition

$$\frac{\partial \phi}{\partial n} + p(x,z)\phi = q(x,z) \quad \text{on } C \tag{109}$$

(II) A Dirichlet and Neumann type boundary condition

$$\left.\begin{aligned} \phi &= p \quad \text{on } s_2 \\ \tfrac{\partial \phi}{\partial n} &= q \quad \text{on } s_1 \end{aligned}\right\} \tag{110}$$

where $c = s_1 + s_2$

For the mixed type boundary condition (109), $p(x,z)$ and $q(x,z)$ are known complex functions. It is to be noted here that for our problem, $p$ and $q$ take the following prescribed values:

$$\left.\begin{aligned} p &= -\tfrac{\varepsilon^2}{q} \quad , \quad q = 0 && \text{along } C_F \text{ of } C \\ p &= 0 \quad , \quad q = -\tfrac{\partial \phi}{\partial n} && \text{on } C_B \text{ of } C \\ p &= 0 \quad , \quad q = 0 && \text{along } C_D \text{ of } C \\ p &= \pm i k_x \quad , \quad q = 0 && \text{along } C_R \text{ of } C \end{aligned}\right\} \tag{111}$$

However, for the Dirichlet and Neumann type boundary condition (110), $p$ and $q$ values are prescribed as follows:

$$\left.\begin{array}{lll} \phi = p = \frac{1}{2}e^{ik_x X} & \text{along } C_F \text{ of } C & \\ \phi = \frac{1}{2}\frac{\cosh k(z+h)}{\cosh kh}e^{ik_x X_R} & \text{along } C_R \text{ of } C \quad \text{or } x = \pm X_R & \\ \frac{\partial\phi}{\partial n} = q = -\frac{\partial\phi_I}{\partial n} & \text{on } C_B \text{ of } C & \\ \frac{\partial\phi}{\partial n} = 0 & \text{along } C_D \text{of } C \quad \text{or } z = -h & \end{array}\right\} \tag{112}$$

It is to be noted that $p$'s and $q$'s of (109) are not the same as those of (110). The $\frac{1}{2}$ is considered to be the reflection or transmission coefficient (see Bai (1977)).

In the following, we describe the boundary element formulations corresponding to these two types of boundary conditions:

## Mixed Type Boundary Conditions

The boundary value problem (108) and (109) can be equivalently posed as the weighted residual equation by introducing a weighting function $\phi^*$ as follows:

$$\int\int_R (\nabla^2 - k_y^2)\phi\phi^* dx dz = \int_C (\frac{\partial\phi}{\partial n} + p\phi - q)\phi^* ds \tag{113}$$

Using Green's Theorem:

$$\begin{aligned} &\int\int_R (\phi^*\nabla^2\phi - \phi\nabla^2\phi^*)dxdz \\ &\quad = \int_C (\phi^*\frac{\partial\phi}{\partial n} - \phi\frac{\partial\phi^*}{\partial n})ds \end{aligned} \tag{114}$$

Equation (113) reduces to

$$\begin{aligned} &-\int\int_R \phi(\nabla^2\phi^* - k_y^2\phi^*)dxdz \\ &\quad + \int_C (\frac{\partial\phi^*}{\partial n} + p\,\phi^*)ds = \int_C q\,\phi^* ds \end{aligned} \tag{115}$$

Assuming that

$$(\nabla^2 - k_y^2)\phi^* = -\Delta^i \tag{116}$$

where $\Delta^i$ is Dirac's delta function at position $i$, then

$$\begin{aligned} \int\int_R \phi(\nabla^2 - k_y^2)\phi dx dz &= -\int\int_R \phi\Delta^i dx dz \\ &= -\phi^i \end{aligned} \tag{117}$$

and (115) may be written as

$$\phi^i + \int_C \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds = \int_C q\,\phi^* ds \tag{118}$$

The fundamental solution of (116) is simply

$$\phi^* = \frac{1}{2\pi} K_0(k_y r) \tag{119}$$

where $K_0(k_y r)$ is the modified Bessel function of second kind and order zero, and

$r = \sqrt{(x - x_i)^2 + (z - z_i)^2}$

### 3.2.1 EQUATION ON THE BOUNDARY

To formulate the problem using the boundary element method, we need to apply (118) to the boundary. This can be done by considering the following semi-circular arc in Figure 4.

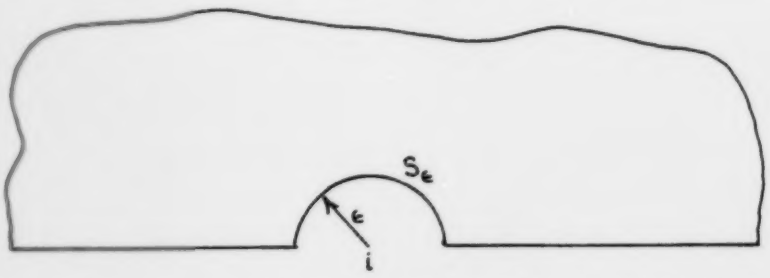



Figure 4: Boundary Arc Assumed to be Semi Circle

The boundary point is assumed to be at the centre of the semi-circule of radius $\epsilon$ whose boundary $s_\epsilon$ forms part of the boundary $C$. In the limit $\epsilon \to 0$, $i$ becomes a boundary point. Therefore, in terms of (118),

$$\begin{aligned}&\int_C \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds \\ &\quad = \int_{C-s_\epsilon} \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds + \int_{s_\epsilon} \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds \end{aligned} \tag{120}$$

It is worth mentioning here that for very small argument of $K_0(k_y r)$, $\phi^*$ simply becomes

$$\phi^* = -\frac{1}{2\pi}\ell n(r) \tag{121}$$

such that

$$\left.\begin{aligned} \phi^*|_{r=\epsilon} &= -\tfrac{1}{2\pi}\ell n\,\epsilon \\ \text{and}\quad \tfrac{\partial \phi^*}{\partial n}|_{r=\epsilon} &= -\tfrac{1}{2\pi\epsilon} \end{aligned}\right\} \tag{122}$$

Therefore,

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{s_\epsilon} \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds \\ &\quad = \lim_{\epsilon\to 0}\int_{s_\epsilon} \phi\{-\frac{1}{2\pi\epsilon} + p(-\frac{1}{2\pi}\ell n\epsilon)\}ds \\ &\quad = \lim_{\epsilon\to 0}[\phi^i\{(-\frac{1}{2\pi\epsilon}(\pi\epsilon)) - \frac{p}{2\pi}\ell n\,\epsilon(\pi\epsilon)\}] \\ &\quad = -\frac{1}{2}\phi^i. \end{aligned} \tag{123}$$

and so

$$\int_C \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*) \text{ contributes to } \int_C \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds - \frac{1}{2}\phi^i \tag{124}$$

Thus when $i$ is on the boundary $C$, we obtain the relation

$$\frac{1}{2}\phi^i + \int_C \phi(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds = \int_C q\,\phi^* ds \tag{125}$$

### 3.2.2 BOUNDARY ELEMENTS

Consider the domain as shown in Figure 5.

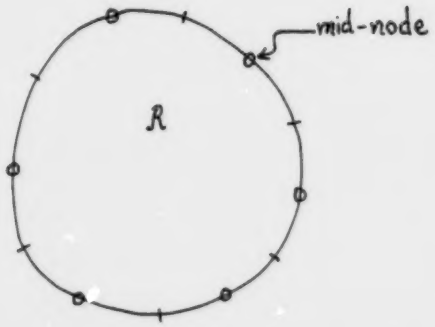



Figure 5: Constant Boundary Elements.

We discretize the boundary into $n$ straight line segments called elements. Nodal points are defined at the middle of each element. The values of $\phi$ and $\frac{\partial \phi}{\partial n}$ are assumed constant on each element and equal to their values at each node.

Therefore, for a particular node $i$, (125) has the discretized form

$$\frac{1}{2}\phi^i + \sum_{j=1}^{n} \phi^j \int_{C_j} (\frac{\partial \phi^*}{\partial n} + p\,\phi^*) ds = \sum_{j=1} q_j \int_{C_j} \phi^* ds \tag{126}$$

We can relate the ith node to the integral over the jth element by defining the variables

$$\left.\begin{array}{rcl} \int_{C_j} \frac{\partial \phi^*}{\partial n} ds & = & \hat{H}_{ij} \\ \int_{C_j} \phi^* ds & = & G_{ij} \end{array}\right\} \tag{127}$$

which imply of (126),

$$\frac{1}{2}\phi^i + \sum_{j=1}^{n} \phi^j (\hat{H}_{ij} + P_j G_{ij}) = \sum_{j=1}^{n} q_j G_{ij}. \tag{128}$$

As integrals in (127) are easily evaluated analytically for constant elements and more complicated for higher order elements, they are numerically computed for all elements, except the ith corresponding to the node under consideration. The value of $\phi$ at $i$ is related to $\phi$

and $\frac{\partial \phi}{\partial n}$ at all nodes (i.e. $\frac{\partial \phi}{\partial n} = q - p\phi$) on the boundary, including $i$. Equation (128) can now be put into the complex linear algebraic form as

$$AX = B \tag{129}$$

where $X$ is the vector of unknown $\phi$ at the boundary. Then the unknown $\frac{\partial \phi}{\partial n}$ at the boundary can be determined from

$$\frac{\partial \phi}{\partial n} = q - p\,\phi \tag{130}$$

Once the values of $\phi$ on the whole boundary are known, we can calculate the value of $\phi$ at any interior point using (118) which, after discretization, becomes

$$\phi^i = \sum_{j=1}^{n} q_j G_{ij} - \sum_{j=1}^{n} \phi_j (\hat{H}_{ij} + p_j G_{ij}) \tag{131}$$

Velocity components $U$ and $W$ at any interior point may be calculated by differential equation (118)

$$U = \frac{\partial \phi^i}{\partial x} = \int_C \frac{\partial}{\partial x}(q\,\phi^*)ds - \int_C \phi \frac{\partial}{\partial x}(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds \tag{132}$$

$$W = \frac{\partial \phi^i}{\partial z} = \int_C \frac{\partial}{\partial z}(q\,\phi^*)ds - \int_C \phi \frac{\partial}{\partial z}(\frac{\partial \phi^*}{\partial n} + p\,\phi^*)ds \tag{133}$$

and the stream function $\psi$ (for axisymmetric flow) can be obtained from

$$\psi(x,z) = \int^z U\,dz = -\int^x W\,dx. \tag{134}$$

## Dirichlet and Neumann Type Boundary Conditions

In this case, the boundary value problem (108) and (110) can be equivalently described by introducing a weighting function $\phi^*$ having continuous first derivative in $R$. As mentioned in the last section, $\phi^*$ turns out to be a solution of the two dimensional Helmholtz equation.

The problem may be posed as the weighted residual equation

$$\int\int_R (\nabla^2 - k_y^2)\phi^* dx\, dz = \int_{s_1} (\frac{\partial \phi}{\partial n} - q)\phi^* ds - \int_{s_2} (\phi - p)\frac{\partial \phi^*}{\partial n} ds \tag{135}$$

Using Green's tneorem (114), Equation (135) can be reduced to

$$\begin{aligned}\int\int_R \phi(\nabla^2 - k_y^2)\phi^* dx\, dz \;\; = \;\; & -\int_{s_1} q\,\phi^* ds + \int_{s_2} p\frac{\partial \phi^*}{\partial n} ds \\ & -\int_{s_2} \phi^* \frac{\partial \phi}{\partial n} ds + \int_{s_1} \phi \frac{\partial \phi^*}{\partial n} ds \end{aligned} \tag{136}$$

where $c = s_1 + s_2$.

Assuming that

$$(\nabla^2 - k_y^2)\phi^* = -\Delta^i \tag{137}$$

where $\Delta^i$ is Dirac's delta function at position $i$, then

$$\int\int_R \phi(\nabla^2 - k_y^2)\phi^* dx\, dz = -\phi^i \tag{138}$$

and (136) can be written as

$$\phi^i + \int_{s_1} \phi \frac{\partial \phi^*}{\partial n} ds + \int_{s_2} p\frac{\partial \phi^*}{\partial n} ds = \int_{s_1} q\,\phi^* ds + \int_{s_2} \frac{\partial \phi}{\partial n}\phi^* ds \tag{139}$$

which can subsequently be written as

$$\phi^i + \int_c \phi(\frac{\partial \phi^*}{\partial n}) ds = \int (\frac{\partial \phi}{\partial n})\phi^* ds \tag{140}$$

This equation is valid inside the fluid domain. As before, the fundamental solution of the Helmholtz equation (137) is simply

$$\phi^* = \frac{1}{2\pi} K_0(k_y r) \tag{141}$$

where $K_0(k_y r)$ is the modified Bessel function of second kind and order zero. Referring to Figure 4, and following the similar analysis, without going into detailed calculations, we can immediately write the equation valid on the boundary as

$$\frac{1}{2}\phi^i + \int_c \phi(\frac{\partial \phi^*}{\partial n})ds = \int_c (\frac{\partial \phi}{\partial n})\phi^* ds \tag{142}$$

Now, referring to Figure 5, we discretize the boundary into $n$ straight line segments called elements. We are considering here constant boundary elements and so the nodal points are defined at the middle point of each element. It is worth mentioning here that out of $n$ elements, $n_1$ belong to $s_1$ and $n_2$ to $s_2$. The values of $\phi$ and $\frac{\partial \phi}{\partial n}$ are assumed constant on each element and equal to their values at each node.

Therefore, for a particular node $i$, (142) has the discretized form

$$\frac{1}{2}\phi^i + \sum_{j=1}^{n} \phi^j \int_{s_j} \frac{\partial \phi^*}{\partial n} ds = \sum_{j=1}^{n} (\frac{\partial \phi}{\partial n}) \int_{s_j} \phi^* ds \tag{143}$$

We can relate the ith node to the integral over the jth element by defining variables

$$\left.\begin{aligned} \int_{s_j} \frac{\partial \phi^*}{\partial n} ds &= \hat{H}_{ij} \\ \int_{s_j} \phi^* ds &= G_{ij} \end{aligned}\right\} \tag{144}$$

which imply of (143),

$$\frac{1}{2}\phi^i + \sum_{j=1}^{n} \phi^j \hat{H}_{ij} = \sum_{j=1}^{n} (\frac{\partial \phi}{\partial n})_j G_{ij}. \tag{145}$$

Integrals (144) are numerically evaluated for all elements, except the ith corresponding to the node under consideration. The value of $\phi$ at $i$ is related to $\phi$ and $\frac{\partial \phi}{\partial n}$ at all nodes on the boundary including $i$. Defining

$$\left.\begin{aligned} H_{ij} &= \hat{H}_{ij} && \text{when } i \neq j \\ H_{ij} &= \hat{H}_{ij} + \tfrac{1}{2} && \text{when } i = j \end{aligned}\right\}, \tag{146}$$

we obtain the linear algebraic form

$$\sum_{j=1}^{n} H_{ij}\phi^j = \sum_{j=1}^{n} G_{ij}(\frac{\partial\phi}{\partial n})_j \tag{147}$$

or, in matrix form,

$$HV = GQ. \tag{148}$$

It is to be noted here that (148) is a complex linear algebraic equation in matrix form in which $n_1$ values of $\frac{\partial\phi}{\partial n}$ and $n_2$ values of $\phi$ are known. Hence, we have a set of $n$ unknowns in (148). Reordering the equation in such a way that all the unknowns are on the left hand side, we can write

$$AX = B \tag{149}$$

where $X$ is the vector of unknowns $\phi$ and $\frac{\partial\phi}{\partial n}$.

Once the values of $\phi$ and $\frac{\partial\phi}{\partial n}$ are known on the whole boundary, $\phi$ may be computed at every interior point from the relation,

$$\phi^i = \sum_{j=1}^{n} G_{ij}(\frac{\partial\phi}{\partial n})_j - \sum_{j=1}^{n} \hat{H}_{ij}\phi^j \tag{150}$$

Velocity components $U$ and $W$ at any interior point may be obtained from (140),

$$U = \frac{\partial\phi^i}{\partial x} = \int_c \frac{\partial\phi}{\partial n}(\frac{\partial\phi^*}{\partial x})ds - \int_c \phi\frac{\partial}{\partial x}(\frac{\partial\phi^*}{\partial n})ds \tag{151}$$

$$W = \frac{\partial\phi^i}{\partial z} = \int_c \frac{\partial\phi}{\partial n}(\frac{\partial\phi^*}{\partial z})ds - \int_c \phi\frac{\partial}{\partial z}(\frac{\partial\phi^*}{\partial n})ds \tag{152}$$

where for our problem,

$$\frac{\partial \phi^*}{\partial x} = -\frac{k_y(x-x_i)}{2\pi r} K_1(k_y r)$$

$$\frac{\partial \phi^*}{\partial z} = -\frac{k_y(z-z_i)}{2\pi r} K_1(k_y r)$$

$$\text{and} \qquad \frac{\partial \phi^*}{\partial n} = -\frac{k_y}{2\pi r} K_1(k_y r)\{(x-x_i)n_x + (z-z_i)n_z\} \tag{153}$$

$$\left.\begin{aligned} n_x &= \cos\beta \\ n_z &= \sin\beta \end{aligned}\right\} \text{ direction cosines.}$$

Here $K_1(k_y r)$ is the modified Bessel function of second kind and order one.

The values of $\frac{\partial}{\partial x}(\frac{\partial \phi^*}{\partial n})$ and $\frac{\partial}{\partial z}(\frac{\partial \phi^*}{\partial n})$ are obtained as

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial \phi^*}{\partial n}\right) &= -\frac{k_y}{2\pi r} K_1(k_y r) n_x + \frac{k_y(x-x_i)}{2\pi r^3} K_1(k_y r)\{(x-x_i)n_x + (z-z_i)n_z\} \\ &\quad -\frac{k_y^2(x-x_i)}{2\pi r^2} K_1'(k_y r)\{(x-x_i)n_x + (z-z_i)n_z\} \end{aligned} \tag{154}$$

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{\partial \phi^*}{\partial n}\right) &= -\frac{k_y}{2\pi r} K_1(k_y r) n_z + \frac{k_y(z-z_i)}{2\pi r^3} K_1(k_y r)\{(x-x_i)n_x + (z-z_i)n_z\} \\ &\quad -\frac{k_y^2(z-z_i)}{2\pi r^2} K_1'(k_y r)\{(x-x_i)n_x + (z-z_i)n_z\} \end{aligned} \tag{155}$$

Here a prime denotes the differentiation with respect to the argument. Using a recurrence relation of Bessel's function, we have

$$K_1'(k_y r) = -K_.(k_y r) - \frac{K_1(k_y r)}{(k_y r)} \tag{156}$$

## 3.3 NUMERICAL COMPUTATIONS

The integrals can be computed using 4-point Gauss quadrature rule for all segments except the one corresponding to the node under consideration. For this particular case, $\hat{H}_{ii}$ and $G_{ii}$ integrals can be calculated analytically.

We have already seen that when $r \to 0$, $K_0(k_y r)$ behaves like $-\ell nr$. Therefore

$$\hat{H}_{ii} = \int_{s_i} \frac{\partial \phi^*}{\partial n} ds = -\frac{1}{2\pi} \int_{s_i} \frac{\partial}{\partial n}(\ell nr) ds \tag{157}$$

and

$$G_{ii} = \int_{s_i} \phi^* ds = -\frac{1}{2\pi} \int_{s_i} (\ell nr) ds \tag{158}$$

Consider the following linear element which contains the point $i$.

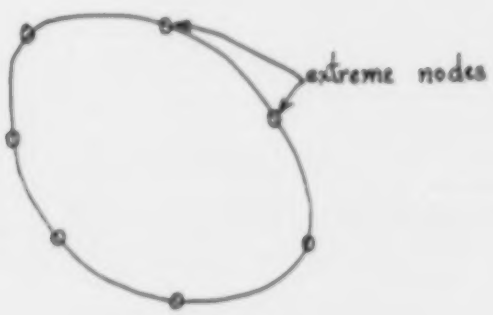



Figure 6: Linear Element Coordinates

It can be easily seen from Figure 6 that

$$\frac{\partial}{\partial n}(\ell nr) = \frac{1}{r}(\frac{\partial r}{\partial n}) = \frac{1}{r}\cos\frac{\pi}{2} = 0.$$

and thus,

$$\hat{H}_{ii} = 0. \tag{159}$$

Consider the following transformation to evaluate $G_{ii}$:

$$r = |r_1|\xi$$

$$dr = |r_1|d\xi$$

Then

$$\begin{aligned} G_{ii} &= -\frac{1}{2\pi}\int_{s_i}(\ell nr)ds \\ &= -\frac{1}{2\pi}\int_{(1)}^{(2)}\ell nr\,dr \\ &= -\frac{1}{\pi}\int_{(0)}^{(2)}\ell nr\,dr \\ &= -\frac{1}{\pi}|r_1|\int_0^1(\ell n|r_1|+\ell n\xi)ds \\ &= -\frac{|r_1|}{\pi}[\ell n|r_1|-1] \\ G_{ii} &= \frac{1}{\pi}|r_1|[\ell n\frac{1}{|r_1|}+1] \end{aligned} \tag{160}$$

In this section, we have investigated the boundary element analysis for the diffraction caused by the wave-ice interaction using the Dirichlet and Neumann type boundary conditions. The flow geometry is discretized as shown in Figure 7 below.

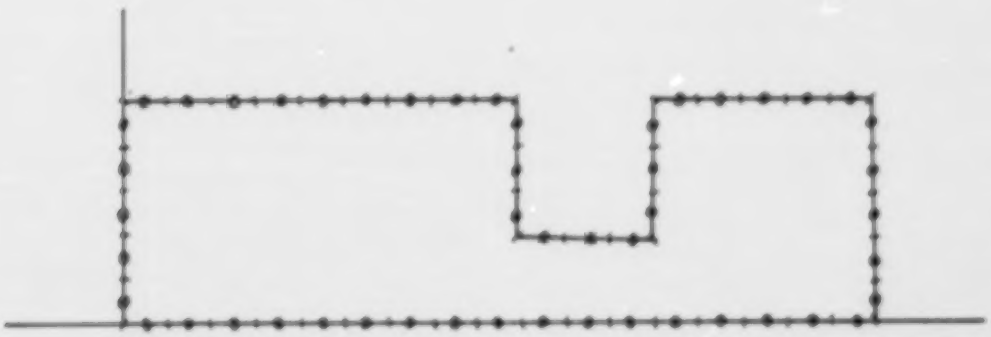


Figure 7: Discretization of the Flow Boundary by Constant Boundary Elements.

We investigate this problem using 6 sets of numerical data as given below. We consider the

depth of the ocean $h = 6$ meters and the halfwidth of the ice flow $a = 1$ metre and therefore $\frac{h}{a} = 6$.

(I) Let $ka = 0.1$, $ka$, diffraction parameter.

Then $kh = 0.6$, $kh$, relative depth.

Thus $k = .1$

If we consider the angle of wave propagation $\alpha = 30°$, then

$$k_x = k\cos\alpha = (.1)(.866) = 0.0866$$

$$k_y = k\sin\alpha = (.1)(.5) = 0.0500.$$

The nondimensional wave number, $\frac{\sigma^2 a}{g} = (ka)\tanh kh = 0.0537$.

(II)

$$\alpha = 45°$$

$$ka = 0.1$$

$$kh = 0.6$$

$$k = 0.1$$

$$k_x = k\cos\alpha = 0.0707$$

$$k_y = k\sin\alpha = 0.0707$$

$$\frac{\sigma^2 a}{g} = 0.0537$$

(III)

$$ka = 0.2 \quad , \quad k = 0.2$$

$$kh = 1.2$$

$$\alpha = 30^\circ$$

$$k_x = k\cos\alpha = 0.1732$$

$$k_y = k\sin\alpha = 0.1000$$

$$\frac{\sigma^2 a}{g} = (ka)\tanh(kh) = 0.1667$$

(IV)

$$ka = 0.2$$

$$k = 0.2$$

$$kh = 1.2$$

$$\alpha = 45^\circ$$

$$k_x = k\cos\alpha = 0.1414$$

$$k_y = k\sin\alpha = 0.1414$$

$$\frac{\sigma^2 a}{g} = 0.1667$$

(V)

$$ka = 0.4$$

$$k = 0.4$$

$$kh = 2.4$$

$$\alpha = 30^\circ$$

$$k_x = k\cos\alpha = 0.3464$$

$$k_y = k\sin\alpha = 0.2000$$

$$\frac{\sigma^2 a}{g} = (ka)\tanh kh = 0.3935$$

(VI)

$$
\begin{aligned}
ka &= 0.4 \\
k &= 0.4 \\
kh &= 2.4 \\
\alpha &= 45^{\circ} \\
k_x &= k\cos\alpha = 0.2828 \\
k_y &= k\sin\alpha = 0.2828 \\
\frac{\sigma^2 a}{g} &= (ka)\tanh kh = 0.3935
\end{aligned}
$$

## 3.4 PHYSICAL QUANTITIES

Once the velocity potential is obtained, we can compute a number of physical quantities of interest. The wave elevation can be obtained by the relation

$$
\begin{aligned}
\eta(x,y,t) &= -\frac{1}{g}(\frac{\partial \Phi}{\partial t})_{z=0} \\
&= +\frac{1}{g} Re[i\sigma\,\phi(x,0)\exp\{i(k_y y - \sigma t)\}]
\end{aligned}
$$

where

$$
\phi(x,0) = \phi_I(x,0) + \phi_S(x,0) \tag{161}
$$

The linearized pressure field is obtained from the relation

$$
\begin{aligned}
p &= -\rho(\frac{\partial \Phi}{\partial t}) \\
&= \rho Re[i\sigma\,\phi(x,z)\exp\{i(k_y y - \sigma t)\}] \qquad (162)
\end{aligned}
$$

The local force and moment acting on a cross-section of the ice floe are, respectively

$$
\vec{F} = \int_{CB} p\,\vec{n}\,ds \tag{163}
$$

$$\vec{M} = \int_{CB} p(\vec{r} \times \vec{n}) ds \tag{164}$$

where

$$\vec{F} = (F_x, F_z)$$

$$\vec{r} = (x, z) \quad \text{the position vector}$$

and

$$\vec{n} = (n_1, n_2)$$

is the unit normal vector into the ice-floe.

The velocity vectors on the vertical cross sectional plane can be obtained from

$$\left.\begin{aligned} V &= \frac{\partial \phi}{\partial x} \\ W &= \frac{\partial \phi}{\partial z} \end{aligned}\right\} \tag{165}$$

## 3.5 TWO SPECIAL CASES

(i) When

$$\alpha = 0$$

$$k_x = k \cos \alpha = k$$

$$k_y = k \sin \alpha = 0.$$

The velocity potential can be written as follows:

$$\begin{aligned} \Phi &= Re[\phi(x,z) e^{i(k_y y - \sigma t)}] \\ &= Re[(\phi_i(x,z) + \phi_s(x,z)) e^{-i\sigma t}] \end{aligned} \tag{166}$$

and consequently the differential equation

$$\frac{\partial^2 \phi_s}{\partial x^2} + \frac{\partial^2 \phi_s}{\partial z^2} - k_y^2 \phi_s = 0 \tag{167}$$

can be reduced to Laplace's equation in two dimension:

$$\frac{\partial^2 \phi_s}{\partial x^2} + \frac{\partial^2 \phi_s}{\partial z^2} = 0. \tag{168}$$

The fundamental solution can then be obtained solving the differential equation

$$\frac{\partial^2 \phi_s^*}{\partial x^2} + \frac{\partial^2 \phi_s^*}{\partial z^2} = -\Delta^i. \tag{169}$$

The fundamental solution is simply

$$\phi_s^* = -\frac{1}{2\pi} \ell n\, r. \tag{170}$$

The boundary condition and the numerical data are given below.

(ii) When

$$\begin{aligned} \alpha &= \frac{\pi}{2} \\ k_x &= k \cos \alpha = 0 \\ k_y &= k \sin \alpha = k \end{aligned}$$

In this case, the velocity potential is given by

$$\begin{aligned} \Phi &= Re[\phi(z) e^{i(k_y y - \sigma t)}] \\ &= Re[(\phi_I(z) + \phi_s(z)) e^{i(k_y y - \sigma t)}] \end{aligned} \tag{171}$$

The differential equation will be $x$ independent and can be written as

$$\frac{\partial^2 \phi_s}{\partial z^2} - k_y^2 \phi_s = 0 \tag{172}$$

The solution of this equation is simply

$$\phi_s = \frac{gA}{\sigma} \frac{\cosh k(z+h)}{\cosh kh} \tag{173}$$

where $\sigma^2 = gk \tanh kh$.

Thus the total velocity potential is

$$\Phi = Re[(\phi_I(z) + \phi_s(s)]e^{i(k_x x - \sigma t)}] \tag{174}$$

where

$$\phi_I = \frac{gA}{\sigma} \frac{\cosh k(z+h)}{\cosh kh} \tag{175}$$

Numerical data:

Let

$$h = 5 \text{ meters}$$

$$a = 1 \text{ meter}$$

Then

$$\frac{h}{a} = 5$$

Let

$$ka = 0.1$$

Then

$$kh = 0.5$$

$$k_x = k = 0.1$$

$$\frac{\sigma^2}{g} = k \tanh kh = 0.0462$$

Boundary conditions:

$$(1)\quad \frac{\partial\phi}{\partial n} - 0.0462\phi = 0 \qquad \text{along } C_F$$

$$(2)\quad \frac{\partial\phi}{\partial n} = 0 \qquad \text{along } C_R$$

$$(3)\quad \frac{\partial\phi}{\partial n} = 0 \qquad \text{along } C_D$$

$$(3)\quad \frac{\partial\phi}{\partial n} = -\frac{\partial\phi_I}{\partial n} \qquad \text{along } C_B$$

$$\phi_I = \frac{\cosh k(z+h)}{\cosh kh} e^{ik_x x}$$

# 4 INTERACTIONS OF WAVES AND WAVE-INDUCED CURRENTS IN A VISCOUS ROTATING OCEAN

The current near the sea surface is an important oceanographic parameter because it contributes to the environmental loading on offshore structures, affects the dispersion and spreading of oil and other pollutants, influences biological processes, such as the drift of fish eggs and larvae, and is prominent in the wave-ice coupling. On medium to long time scales, it can also be expected to have some influence on the weather and climate, as well as the circulation of the deeper ocean, in association with other sea surface processes such as turbulence and surface waves.

Ekman (1905) demonstrated the effect the earth's rotation has on vertical current structure. If the vertical momentum transport (by turbulent mixing, for example) is represented by an eddy viscosity $v$, the direct shearing effect of the wind is limited to a depth of $\sim \sqrt{2v/f}$ where $f$ is the Coriolis parameter. Significant interior implications are generated by variations in horizontal wind forcing causing vertical water movements via continuity and denoted

'Ekman pumping' (Gill 1982).

If $v$ is constant, the current evolves to a steady state which is about 45° to the right of the wind direction in the Northern Hemisphere, decreases exponentially and rotates clockwise with depth. However, Huang (1979) reports that field observations of surface drift currents indicate the wind to current angle to be about 10°. Indeed, small angular deviations like this are simulated if we use a vertically varying eddy viscosity which increases rapidly with depth near the surface following Madsen (1977). Weber (1983) found that a constant eddy viscosity could reduce the wind-to-current angle from 45° if the effect of wind-generated surface waves was included.

We concentrate on the effects of surface waves in this section, and its coupling to surface currents following Jenkins (1991). The rate of momentum transfer between the atmosphere and the ocean, described by the "wind stress" depends on surface roughness which is related to the surface wave field. Wave breaking and other forms of wave energy dissipation insert turbulent kinetic energy into the near surface ocean and thus contribute to near-surface mixing. Other contributions to near-surface mixing include Langmuir circulations, which may probably be induced by instabilities in the wind and wave-induced current field.

## 4.1 MATHEMATICAL FORMULATION

We consider a homogeneous incompressible viscous ocean rotating about the vertical axis with a constant angular velocity $\frac{1}{2}\Omega$, where $\Omega$ is the Coriolis parameter. It is assumed that the depth of the ocean is infinite and the ocean surface is unlimited. The surface is horizontal when there are no ocean waves. A Cartesian coordinate system is considered with the $(x, y)$ axes situated at the undisturbed ocean surface and the $z$ axis is positive upwards. The

motion is described by Lagrangian rather than Eulerian coordinates. A fluid particle $(a, b, c)$ having initial coordinates $(x_0, y_0, z_0)$ occupies the position $(x, y, z)$ at a later time $t$. The velocity components and accelerations are given by $(x_t, y_t, z_t)$ and $(x_{tt}, y_{tt}, z_{tt})$, respectively. Here the subscripts denote the partial differentiation. Following the work of Lamb (1932) and Pierson (1962), the equation of continuity for incompressible flow in Lagrangian form is given by

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = \frac{\partial(x_0, y_0, z_0)}{\partial(a, b, c)}. \tag{176}$$

It is noted that if the fluid particle $(a, b, c)$ starts from the initial coordinates $(x_0, y_0, z_0)$ then $\frac{\partial(x_0, y_0, z_0)}{\partial(a, b, c)} = 1$ and so (176) can be written as

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = 1. \tag{177}$$

By inclusion of rotation for incompressible flow, the Navier-Stokes equations in Eulerian formulation are given by

$$\begin{aligned} x_{tt} - \Omega y_t &= -\frac{P_x}{\rho} + \nu \nabla^2 x_t \\ y_{tt} + \Omega x_t &= -\frac{P_y}{\rho} + \nu \nabla^2 y_t \\ z_{tt} + g &= -\frac{P_z}{\rho} + \nu \nabla^2 z_t \end{aligned} \tag{178}$$

where $x_{tt} = U_t + UU_x + VU_y + WU_z$ and $\nabla^2 U = U_{xx} + U_{yy} + U_{zz}$ etc, $P$ is the pressure, $\rho$ is the density, $\nu$ is the constant coefficient of kinematic viscosity and $g$, the acceleration due to gravity.

Using the method suggested by Lamb (1932), these three equations can be rewritten as follows:

$$(x_{tt} - \Omega y_t - \nu \nabla^2 x_t) x_a + (y_{tt} + \Omega x_t - \nu \nabla^2 y_t) y_a + (z_{tt} + g - \nu \nabla^2 z_t) z_a = -\frac{1}{\rho} P_a$$

$$(x_{tt} - \Omega y_t - \nu\nabla^2 x_t)x_a + (y_{tt} + \Omega x_t - \nu\nabla^2 y_t)y_a + (z_{tt} + g - \nu\nabla^2 z_t)z_a = -\frac{1}{\rho}P_a$$

$$(x_{tt} - \Omega y_t - \nu\nabla^2 x_t)x_c + (y_{tt} + \Omega x_t - \nu\nabla^2 y_t)y_c + (z_{tt} + g - \nu\nabla^2 z_t)z_c = -\frac{1}{\rho}P_c \quad (179)$$

Note that $P_a x_a + P_y y_a + P_z z_a = P_a$.

Solving these equations by Cramer's Rule, we obtain

$$\begin{aligned} x_{tt} - \Omega y_t &= -\frac{1}{\rho}\frac{\frac{\partial(P,y,z)}{\partial(a,b,c)}}{\frac{\partial(x,y,z)}{\partial(a,b,c)}} + \nu\nabla^2 x_t \\ y_{tt} - \Omega x_t &= -\frac{1}{\rho}\frac{\frac{\partial(x,P,z)}{\partial(a,b,c)}}{\frac{\partial(x,y,z)}{\partial(a,b,c)}} + \nu\nabla^2 y_t \\ z_{tt} + g &= -\frac{1}{\rho}\frac{\frac{\partial(x,y,P)}{\partial(a,b,c)}}{\frac{\partial(x,y,z)}{\partial(a,b,c)}} + \nu\nabla^2 z_t \end{aligned} \quad (180)$$

Here $\nabla^2$ operator is still in Eulerian form. To obtain the Lagrangian form, we proceed as follows:

$$\begin{aligned} U_a &= U_x x_a + U_y y_a + U_z z_a \\ U_b &= U_x x_b + U_y y_b + U_z z_b \\ U_c &= U_x x_c + U_y y_c + U_z z_c \end{aligned} \quad (181)$$

solving this set in forms of $U_x$, $U_y$ and $U_z$, we obtain

$$\begin{aligned} U_x &= \frac{\partial(u,y,z)}{\partial(a,b,c)} \\ U_y &= \frac{\partial(x,u,z)}{\partial(a,b,c)} \\ U_z &= \frac{\partial(x,y,u)}{\partial(a,b,c)} \end{aligned}$$

and consequently we can write $\nabla^2 U \equiv \nabla^2 x_t$ as

$$\nabla^2(x_t) = \frac{\partial(\frac{\partial(x,y,z)}{\partial(a,b,c)},y,z)}{\partial(a,b,c)} + \frac{\partial(x,\frac{\partial(x,y,z)}{\partial(a,b,c)},z)}{\partial(a,b,c)} + \frac{\partial(x,y,\frac{\partial(x,y,z)}{\partial(a,b,c)})}{\partial(a,b,c)}. \quad (182)$$

Similar expressions would result for $\nabla^2 y_i$ and $\nabla^2 z_i$. Considering small perturbations from the basic state, $(a, b, c)$, we write the perturbation solutions (see Pierson (1962)):

$$\left.\begin{aligned} x &= a + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \\ y &= b + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots \\ z &= c + \epsilon z_1 + \epsilon^2 z_2 + \epsilon^3 z_3 + \cdots \\ P &= P_0 - \rho g c + \epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 P_3 + \cdots \end{aligned}\right\} \tag{183}$$

Here the expansion parameter $\epsilon$ is essentially proportional to the amplitude of the initial surface wave and will be discussed later. The free surface is given by $c = 0$ for all times. This is a linear kinematic boundary condition as opposed to the equivalent nonlinear Eulerian version. This shows some of the disadvantages by using a Lagrangian description for problems involving a freely moving boundary. In Eulerian form, the free surface is given by $z = \eta$. Using the perturbation scheme (183) the appropriate Lagrangian description of the surface becomes

$$\eta = \epsilon z_1 + \epsilon^2 z_2 + \epsilon^3 z_3 + \cdots \quad (c = 0). \tag{184}$$

As this study is concerned with the infinitely deep ocean, all perturbation quantities are assumed to vanish when $c \to -\infty$.

When (183) is substituted into the continuity equation (176) and the equations of motion (180) and when terms in equal powers of $\epsilon$ are collected and set equal to each other, the zero-order terms balance out exactly. The first order terms are given by

$$\left.\begin{aligned} & x_{1a} + y_{1b} + z_{1c} = 0 \\ & x_{1tt} - \Omega y_{1t} + g x_{1a} + \frac{P_{1a}}{\rho} = \nu \nabla_1^2 x_{1t} \\ & y_{1tt} + \Omega x_{1t} + g z_{1b} + \frac{P_{1b}}{\rho} = \nu \nabla_1^2 y_{1t} \\ & z_{1tt} + g z_{1c} + \frac{P_{1c}}{\rho} = \nu \nabla_1^2 z_{1t} \end{aligned}\right\} \tag{185}$$

where $\nabla_L^2 \equiv \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial c^2}$. The second-order equations are:

$$
\begin{aligned}
x_{2a} + y_{2b} + z_{2c} + x_{1a}y_{1c} + x_{1a}z_{1c} \;+\; & y_{1b}z_{1c} - x_{1c}z_{1a} - y_{1c}z_{1b} - x_{1b}y_{1a} = 0 \\
x_{2tt} - \Omega y_{2t} + g z_{2a} + \frac{1}{\rho}P_{2a} - \nu\nabla_L^2 x_{2t} \;=\; & \frac{1}{\rho}(P_{1a}x_{1a} + P_{1b}y_{1a} + P_{1c}z_{1a}) \\
& - g(z_{1a}y_{1b} - y_{1a}z_{1b}) \\
& - 2\nu(x_{1a}x_{1taa} + y_{1b}x_{1tbb} + z_{1c}x_{1tcc}) \\
& - 2\nu(x_{1tab}(x_{1b} + y_{1a}) + x_{1tac}(x_{1c} + z_{1a}) + x_{1tbc}(y_{1c} + z_{1b})) \\
& - \nu(x_{1ta}\nabla_L^2 x_1 + x_{1tb}\nabla_L^2 y_1 + x_{1tc}\nabla_L^2 z_1) \\
y_{2tt} - \Omega x_{2t} + g z_{2b} + \frac{1}{\rho}P_{2b} - \nu\nabla_L^2 y_{2t} \;=\; & \frac{1}{\rho}(P_{1a}x_{1b} + P_{1b}y_{1b} + P_{1b}z_{1b}) \\
& - g(x_{1b}z_{1a} - x_{1a}z_{1b}) \\
& - 2\nu(x_{1a}y_{1taa} + y_{1b}y_{1tbb} + z_{1c}y_{1tcc}) \\
& - 2\nu(y_{1tab}(x_{1b} + y_{1a}) + y_{1tac}(x_{1c} + z_{1a}) + y_{1tbc}(y_{1c} + z_{1b})) \\
& - \nu(y_{1ta}\nabla_L^2 x_1 + y_{1tb}\nabla_L^2 y_1 + y_{1tc}\nabla_L^2 z_1) \\
z_{2tt} + g z_{2c} + \frac{1}{\rho}P_{2c} - \nu\nabla_L^2 z_{2t} \;=\; & \frac{1}{\rho}(P_{1a}x_{1c} + P_{1b}y_{1c} + P_{1c}z_{1c}) \\
& - g(x_{1a}y_{1b} - x_{1a}y_{1a}) \\
& - 2\nu(x_{1a}z_{1taa} + y_{1b}z_{1tbb} + z_{1c}z_{1tcc}) \\
& - 2\nu(z_{1tab}(x_{1b} + y_{1a}) + z_{1tac}(x_{1c} + z_{1a}) + z_{1tbc}(y_{1c} + z_{1b})) \\
& - \nu(z_{1ta}\nabla_L^2 x_1 + z_{1tb}\nabla_L^2 y_1 + z_{1tc}\nabla_L^2 z_1). \qquad (186)
\end{aligned}
$$

It is noted here that the equations (186) turn out to be linear, and their analytical solutions are possible on the basis of the standard method available for the solution of partial differential equations. The right hand side of each equation is, of course, a known function of $a$,

$b$, $c$, and $t$ once an appropriate solution of equation (185) has been found. In the following we shall demonstrate some simple solutions of physical interest.

## 4.2 VISCOSITY INFLUENCED WAVE MOTION (1D)

The most idealized viscosity dominated wave motion in deep ocean can be obtained from the solution of the equation (see equation (185)):

$$z_{1tt} = \nu \frac{\partial^2}{\partial c^2}(z_{1t}). \tag{187}$$

If we consider that the motion is oscillatory with respect to time and damped oscillatory with respect to the coordinate $c$, then we can choose

$$(z_{1t}) = Re[w(c)e^{-i\sigma t}] \tag{188}$$

such that (187) can be reduced to

$$\frac{d^2w}{dc^2} = -\frac{i\sigma}{\nu}w. \tag{189}$$

The solution can be written as

$$(z_{1t}) = Re[(Ae^{\gamma(1-i)c} + Be^{-\gamma(1-i)c})e^{-i\sigma t}] \tag{190}$$

where

$$\gamma = (\frac{\sigma}{2\nu})^{\frac{1}{2}}, \tag{191}$$

called the viscous length scale. The reciprocal of this parameter is called the vorticity-layer depth $\ell = \gamma^{-1} = (\frac{2\nu}{\sigma})^{\frac{1}{2}})$. The infinite depth boundary condition demands that when $c \to -\infty$, (190) must remain finite.

Therefore, $B$ must be zero because $\gamma$ is a positive quantity. Thus

$$(z_{1t}) = Re[Ae^{\gamma(1-i)c-i\sigma t}]. \tag{192}$$

The surface boundary condition at $c = 0$ is given by

$$(z_{1t})_{c=0} = Re[ae^{-i\sigma t}] \tag{193}$$

and so we have the solution as

$$(z_{1t}) = Re[a e^{\gamma(1-i)c-i\sigma t}] \tag{194}$$

where $a$ is a real amplitude of the wave. Thus (194) can be written as

$$(z_{1t}) = ae^{\gamma c}\cos(\gamma c + \sigma t). \tag{195}$$

The particle position can be obtained as

$$z_1 = \frac{ae^{\gamma c}}{\sigma}\sin(\gamma c + \sigma t) + z_0 \tag{196}$$

where $z_0$ is a constant.

Here $\sigma = (gk)^{\frac{1}{2}}$ is the frequency of small amplitude deep water waves with wave number $k$.

## 4.3 THE VISCOSITY ATTENUATED PRIMARY WAVES

The primary wave can be obtained from the solution of the linearized equations (185). We consider an idealized model of ocean swell, relatively short, high frequency gravity waves. Usually such waves will have wavelength of $\lambda$ of around 100m and frequencies of about $1s^{-1}$. Under this condition we can safely neglect the effect of rotation on the first order solution.

We consider the wave propagating along the x-axis ($\frac{\partial}{\partial b} = 0$ in the perturbation) and

taking $y_1 = 0$, the first order equations (185) then reduces to (after neglecting the rotation)

$$\left.\begin{aligned} x_{1at} + g x_{1a} &= -\tfrac{1}{\rho} P_{1a} + \nu \nabla_L^2 x_{1t} \\ x_{1ct} + g x_{1c} &= -\tfrac{1}{\rho} P_{1c} + \nu \nabla_L^2 x_{1t} \\ x_{1a} + x_{1c} &= 0 \end{aligned}\right\} \tag{197}$$

where

$$\nabla_L^2 = \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial c^2}$$

and

$$\frac{\partial(x_0, y_0, z_0)}{\partial(a, b, c)} = 1 + O(\epsilon^2).$$

It is observed that the solution is $O(\epsilon)$ involves the ratio $\frac{k}{\gamma} \ll 1$ for the primary waves. We introduce the attenuation coefficient $\beta$ by the relation.

$$\beta = 2\nu k^2 = \frac{\sigma k^2}{\gamma^2}. \tag{198}$$

We follow the method initially by Lamb (1932) to solve the equations (197).

Assume the solutions for $x_{1t}$ and $x_{1t}$ exist in the forms:

$$\left.\begin{aligned} x_{1t} &= -\tfrac{\partial \Phi}{\partial a} - \tfrac{\partial \Psi}{\partial c} \\ x_{1t} &= -\tfrac{\partial \Phi}{\partial c} + \tfrac{\partial \Psi}{\partial a} \end{aligned}\right\}. \tag{199}$$

It can be easily seen that the continuity equation $\frac{\partial}{\partial a}(x_{1t}) + \frac{\partial}{\partial c}(x_{1t}) = 0$ becomes

$$\nabla_L^2 \Phi = 0. \tag{200}$$

Differentiating the first equation (197) with respect to $c$ partially and second equation with respect to $a$ and then subtracting one from the other we obtain,

$$\frac{\partial}{\partial c}(x_{1at}) - \frac{\partial}{\partial a}(x_{1ct}) = \nu \nabla_L^2 \left(\frac{\partial x_{1t}}{\partial c} - \frac{\partial x_{1t}}{\partial a}\right)$$

or,

$$\frac{\partial}{\partial t}(\nabla_L^2\Psi) = \nu\nabla_L^2(\nabla_L^2\Psi)$$

which can subsequently be written as

$$\frac{\partial\Psi}{\partial t} = \nu\nabla_L^2\Psi. \tag{201}$$

From Bernoulli's equation we have

$$\frac{P_1}{\rho} = \frac{\partial\Phi}{\partial t} - gz_1. \tag{202}$$

Equations (200) and (201) have solutions of the forms

$$\Phi = Re[Ae^{kc}e^{i(kx+nt)}] \tag{203}$$

$$\Psi = Re[Be^{mc}e^{i(kx+nt)}] \tag{204}$$

where

$$m^2 = k^2 + \frac{n}{\nu}. \tag{205}$$

The first-order surface equation then becomes

$$\begin{aligned} \eta &= \epsilon z_1 \qquad (c=0) \\ &= -\epsilon\frac{k}{n}Re[(A-iB)e^{i(kx+nt)}]. \end{aligned} \tag{206}$$

## 4.4 FREE SURFACE BOUNDARY CONDITIONS

To evaluate the constants $A$ and $B$ we need two conditions and these are obtained by the stress conditions at the free surface. If $T_1$ denotes the surface tension, the stress conditions at the free surface are given by

$$p^{(xx)} = T_1\frac{\partial^2\eta}{\partial x^2} \tag{207}$$

$$p^{(xz)} = 0 \tag{208}$$

where $P^{(zz)}$ and $P^{(xz)}$ are vertical and horizontal stresses respectively at the sloping free surface. These stresses can be expressed as

$$\left.\begin{aligned} P^{(zz)} &= -P + 2\mu\frac{\partial w}{\partial z} \\ P^{(xz)} &= \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \end{aligned}\right\} \tag{209}$$

where $u$ and $w$ are the horizontal and vertical components in Cartesian form. The Lagrangian form of (209) is given by

$$\begin{aligned} P^{(zz)} &= -P + 2\mu(\frac{z_{1a}x_c - z_{1c}x_a}{x_c z_a - x_a z_c}) \\ P^{(xz)} &= \mu(\frac{u_a x_c - x_a u_c) + (w_c z_a - w_a z_c)}{x_c z_a - x_a z_c}) \end{aligned} \tag{210}$$

Using the expansions (183) we can write these stresses as

$$\begin{aligned} P^{(zz)} &= -P_0 + \epsilon[P_1 + 2\mu z_{1tc}] \\ &\quad + \epsilon^2[-P_2 + \mu(2z_{2tc} + 2x_{2tc} + 2x_{1a}z_{1tc} - 2x_{1c}z_{1ta} - x_{1tc}z_{1a} - z_{1ta}z_{1a})] \\ &\quad + O(\epsilon^3) \quad , \quad c = 0 \end{aligned} \tag{211}$$

$$\begin{aligned} P^{(xz)} &= \epsilon\mu[x_{1ct} + z_{1ta}] \\ &\quad + \epsilon^2[P_1 z_{1a} + \mu(x_{2tc} + z_{2tc} + x_{1a}x_{1tc} - x_{1ta}x_{1c} + z_{1ta}z_{1c} - z_{1a}z_{1tc} - 2x_{1ta}z_{1a})] \\ &\quad + O(\epsilon^3) \quad , \quad c = 0 \end{aligned} \tag{212}$$

where $\frac{\mu}{\rho} = \nu$. Thus the boundary conditions (207) and (208) up to order $O(\epsilon)$ can be written as

$$\left.\begin{aligned} -P_1 + 2\mu z_{1tc} &= T_1 z_{1aa}, \qquad c = 0 \\ x_{1tc} + z_{1ta} &= 0 \qquad\qquad\quad c = 0 \end{aligned}\right\}. \tag{213}$$

Thus we have in complex form

$$-\frac{P_1}{\rho} + 2\nu z_{1tc} - \frac{T_1}{\rho}z_{1aa} = -\frac{\partial\Phi}{\partial t} + (g + \frac{T_1}{\rho}k^2)z_1 + 2\nu z_{1tc}$$

$$= -\frac{1}{n}[(n^2 + 2\nu k^2 n + gk + \frac{T_1}{\rho}k^3)A$$

$$- i(gk + \frac{T_1}{\rho}k^3 + 2\nu kmn)B] \quad (214)$$

$$\frac{\mu}{\rho}(x_{1tc} + x_{1ta}) = (-2\frac{\partial^2\Phi}{\partial a \partial c} + \frac{\partial^2\Psi}{\partial a^2} - \frac{\partial^2\Psi}{\partial c^2})\frac{\mu}{\rho}$$

$$= -(2i\nu k^2 A + (n + 2\nu k^2)B). \quad (215)$$

Therefore, the boundary conditions (213) can be satisfied if

$$(n^2 + 2\nu k^2 n + gh + \frac{T_1}{\rho}k^3)A - i(gk + \frac{T_1}{\rho}k^3 + 2\nu kmn)B = 0$$

$$2i\nu k^2 A + (n + 2\nu k^2)B = 0 \quad (216)$$

Eliminating $A$ and $B$, we obtain

$$(n^2 + 2\nu k^2 n + gk + \frac{T_1}{\rho}k^3)(n + 2\nu k^2) = (2\nu k^2)(gk + \frac{T_1}{\rho}k^3 + 2\nu kmn)$$

putting

$$n + 2\nu k^2 = x\sigma$$

$$gk + \frac{T_1}{\rho}k^3 = \sigma^2$$

$$\nu\frac{k^2}{\sigma} = \theta$$

the above equation can be written as

$$(x^2 + 1)^2 = 16\theta^3(x - \theta). \quad (217)$$

From physical considerations we can safely assume that $\theta$ is very small and so $x = \pm i$ thus

$$n = -2\nu k^2 \pm i\sigma. \quad (218)$$

Consider that the wave is propagating along the positive $x$ axis, then

$$n = -2\nu k^2 - i\sigma. \quad (219)$$

Therefore, we can write the physical quantities as follows,

$$\begin{aligned}\Phi &= Re[Ae^{kc}e^{i(ka-\sigma t)}e^{-2\nu k^2 t}] \\ \Psi &= Re[Be^{mc}e^{i(ka-\sigma t)}e^{-2\nu k^2 t}].\end{aligned} \tag{220}$$

Now, defining the attenuation coefficient as $\beta = 2\nu k^2$ and assuming that $\frac{k}{\gamma} \ll 1$ which is true for this kind of problem, we estimate that

$$\begin{aligned}m^2 &= k^2 + \frac{n}{\nu} \\ &= -k^2 - i\frac{\sigma}{\nu} \approx -i\frac{\sigma}{\nu}\end{aligned}$$

such that

$$m \approx \pm(1-i)\gamma. \tag{221}$$

From equation (216), we have

$$\frac{B}{A} = -\frac{2i\nu k^2}{n + 2\nu k^2} = \frac{+2\nu k^2}{\sigma}. \tag{222}$$

Without loss of generality, we choose $A = 1$ so that

$$B = \frac{2\nu k^2}{\sigma}$$

Therefore, equations (220) can be written as

$$\begin{aligned}\Phi &= e^{-\beta t}e^{kc}\cos(ka - \sigma t) \\ \Psi &= \frac{2\nu k^2}{\sigma}e^{-\beta t}e^{\gamma c}\cos(ka - \gamma c - \sigma t) \\ \eta &= \frac{\epsilon k}{\sigma}e^{-\beta t}\sin(ka - \sigma t).\end{aligned} \tag{223}$$

Assume that we initially start with a monochromatic wave

$$\eta = \eta_0 \sin(ka - \sigma t). \tag{224}$$

By comparison with (223), we see that our perturbation parameters can be written as

$$\epsilon = \eta_0 \frac{\sigma}{k}. \tag{225}$$

Substituting the values of $\Phi$ and $\Psi$ into equations (199) and then integrating with respect to time, we obtain the following solutions:

$$\begin{aligned} x_1 &= \frac{ke^{kc}}{\sigma^2+\beta^2} e^{-\beta t}[\sigma\cos(ka-\sigma t) - \beta\sin(ka-\sigma t)] \\ &\quad + \frac{\gamma\beta e^{\gamma c}}{\sigma(\sigma^2+\beta^2)} e^{-\beta t}[(\sigma+\beta)\sin(ka-\gamma c-\sigma t) - (\sigma-\beta)\cos(ka-\gamma c-\sigma t)] \quad (226) \\ z_1 &= \frac{ke^{kc}}{\sigma^2+\beta^2} e^{-\beta t}[\beta\cos(ka-\sigma t) + \sigma\sin(ka-\sigma t)] \\ &\quad + \frac{k\beta e^{\gamma c}}{\sigma(\sigma^2+\beta^2)} e^{-\beta t}[\beta\sin(ka-\gamma c-\sigma t) - \sigma\cos(ka-\gamma c-\sigma t)]. \quad (227) \end{aligned}$$

From equation (202), we can obtain the first order pressure field:

$$\begin{aligned} \frac{P_1}{\rho} &= \frac{\partial\Phi}{\partial t} - gz_1 \\ &= e^{kc}e^{-\beta t}[\sigma\sin(ka-\sigma t) - \beta\cos(ka-\sigma t)] - gz_1. \quad (228) \end{aligned}$$

When $\sigma \gg \beta$, these equations can be reduced to

$$\begin{aligned} x_1 &= \frac{k}{\sigma} e^{-\beta t}[[e^{kc} - \frac{k}{\gamma}e^{\gamma c}(\cos\gamma c + \sin\gamma c)]\cos(ka-\sigma t) \\ &\quad + \frac{k}{\gamma}[e^{\gamma c}(\cos\gamma c - \sin\gamma c) - \frac{k}{\gamma}e^{kc}]\sin(ka-\sigma t) + O(\frac{k^3}{\gamma^3})] \quad (229) \\ z_1 &= \frac{k}{\sigma} e^{-\beta t}[[e^{kc} - \frac{k^2}{\gamma^2}e^{\gamma c}\sin\gamma c]\sin(ka-\sigma t) \\ &\quad + \frac{k^2}{\gamma^2}[e^{\gamma c}\cos\gamma c - e^{kc}]\cos(ka-\sigma t) + O(\frac{k^4}{\gamma^4})] \quad (230) \\ P_1 &= \rho\sigma\frac{k^2}{\gamma^2} e^{-\beta t}[(e^{\gamma c}\cos\gamma c - 2e^{kc})\cos(ka-\sigma t) \\ &\quad + e^{\gamma c}\sin\gamma c\sin(ka-\sigma t) + O(\frac{k^2}{\gamma^2})]. \quad (231) \end{aligned}$$

When $\frac{k}{\gamma} \ll 1$ and $\rho = 0$, then we can express the permanent wave solutions as follows:

$$\left.\begin{aligned} x_1 &= \tfrac{k}{\sigma} e^{kc} \cos(ka - \sigma t) + \tfrac{k^2}{\gamma\sigma} e^{\gamma c}[\sin(ka - \gamma c - \sigma t) - \cos(ka - \gamma c - \sigma t)] \\ z_1 &= \tfrac{k}{\sigma} e^{kc} \sin(ka - \sigma t) + \tfrac{k^3}{\gamma^2\sigma} e^{\gamma c} \cos(ka - \gamma c - \sigma t) \\ P_1 &= \tfrac{\rho g k^3}{\gamma^2\sigma} e^{\gamma c} \cos(ka - \gamma c - \sigma t) \end{aligned}\right\} . \qquad (232)$$

The solutions (232) are identical to those of Weber (1983).

## 4.5 THE ATTENUATED SECONDARY MEAN FLOW

Averaging the second order equations (186) over one wavelength, and including rotation, we obtain for the horizontal mean flow

$$\begin{aligned} \bar{x}_{2tt} \;-\; & \Omega\bar{y}_{2t} + \nu\bar{x}_{2tcc} = \frac{1}{\rho}[\overline{P_{1a}x_{1a}} + \overline{P_{1c}z_{1a}}] \\ -\; & \nu[2\overline{x_{1a}x_{1taa}} + 2\overline{z_{1c}x_{1tc}} + 2\overline{x_{1tac}x_{1c}} + 2\overline{x_{1tac}z_{1a}} + \overline{x_{1tac}\nabla_L^2 x_1} + \overline{x_{1tc}\nabla_L^2 z_1}] \qquad (233) \\ \bar{y}_{2tt} \;+\; & \Omega\bar{x}_{2t} - \nu\bar{y}_{2tcc} = 0 \qquad (234) \end{aligned}$$

where the overbar denotes the spatial average. It is to be noted here that we have assumed $y_1 = 0$ and $\frac{\partial}{\partial b} \equiv 0$. Also there is no mean horizontal pressure gradient to $O(\epsilon^2)$. The mean displacement to $O(\epsilon^2)$ of a particle is given by

$$\left.\begin{aligned} \bar{x} &= a + \epsilon^2\bar{x}_2 \\ \bar{y} &= b + \epsilon^2\bar{y}_2 \\ \bar{z} &= c + \epsilon^2\bar{z}_2 \end{aligned}\right\} . \qquad (235)$$

Defining

$$\left.\begin{aligned} U &= \epsilon^2\bar{x}_{2t} \\ V &= \epsilon^2\bar{y}_{2t} \end{aligned}\right\} \qquad (236)$$

as horizontal mean flow components, a complex velocity $W$ can be written as

$$W = U + iV \tag{237}$$

Substituting the values of $x_1$, $z_1$ and $P_1$ from (229), (230) and (231) into (233) and (234), the equation for the mean motion can be written as

$$\nu W_{cc} - W_t - i\Omega W = \nu\eta_0^2\sigma k^3 e^{-2\beta t}[8e^{2kc} - \tfrac{4\gamma}{k}e^{\gamma c}(\cos\gamma c - \sin\gamma c)] \qquad \text{for } \gamma \gg k. \tag{238}$$

The boundary condition to $O(\epsilon^2)$ due to Unluata and Mei (1970), may be written

$$W_c = -2\frac{\eta_0^2\sigma^2}{k^2}\frac{\partial}{\partial t}(\overline{x_{1a}x_{1c}}) \quad \text{at } c = 0 \tag{239}$$

For $\frac{k}{\gamma} \ll 1$, we have $W_c = 0$ at $c = 0$. Also we need that $W = 0$ as $c \to -\infty$. Before we proceed to obtain the solution of (238), we introduce the following depth parameters:

$$\text{Stokes depth} : L = \frac{1}{2k}$$

$$\text{Vorticity layer depth} : \ell = \frac{1}{\gamma} = (\frac{2\nu}{\sigma})^{\frac{1}{2}}$$

$$\text{Ekman depth} : D = (\frac{2\nu}{\Omega})^{\frac{1}{2}}$$

Using the fact that $\frac{\ell^2}{D^2} = \frac{\Omega}{\sigma} \ll 1$, the particular solution of (238) can be written as

$$W^{(P)} = \eta_0^2\sigma k[\frac{e^{2kc}}{1 - i\frac{L^2}{D^2}} - \frac{2ke^{\gamma c}}{\gamma}(\cos\gamma c + \sin\gamma c)]e^{-2\beta t} \tag{240}$$

The solution to the homogeneous problem can be obtained by using the Laplace transform method. The boundary conditions are

$$W_c^{(h)} = -W_c^{(P)} \qquad c = 0 \tag{241}$$

$$W_c^{(h)} \to 0 \qquad \text{as } c \to -\infty \tag{242}$$

where $W^{(P)}$ is given by (240). The initial condition is given by

$$W^{(A)} = 0 \qquad \text{at } t = 0 \tag{243}$$

By using the results and convolution properties of Laplace transforms, we get

$$W^{(A)} = 2\eta_0^2\sigma k^2(2 - \frac{1}{1 - i\frac{L^2}{D^2}})(\frac{\nu}{\pi})^{\frac{1}{2}}e^{-2\Omega t}\int_0^t \frac{e^{((2\Omega - i\Omega)A - \frac{z^2}{4\nu A})}}{A^{\frac{1}{2}}}dA \tag{244}$$

This solution describes the development of an Ekman current subject to stress (240). Putting $\Omega = 0$ and $t_1 = t - A$ we obtain from (236),

$$U = 2\eta_0^2\sigma k^2(\frac{\nu}{\pi})^{\frac{1}{2}}\int_0^t \frac{e^{-2\Omega t_1 - \frac{z^2}{4\nu(t-t_1)}}}{(t-t_1)^{\frac{1}{2}}}dt \tag{245}$$

which is exactly identical to Longuet-Higgins' (1969) solution. For small times the full solution at the surface can be written

$$\begin{aligned} U &= \frac{\eta_0^2\sigma k}{1 + \frac{L^4}{D^4}}[1 + (\frac{D^4 + 2L^4}{D^4})\frac{D}{L}(\frac{2\Omega t}{\pi})^{\frac{1}{2}}] + 0(t) \\ V &= \frac{\eta_0^2\sigma k\frac{L^2}{D^2}}{1 + \frac{L^4}{D^4}}[1 - \frac{D}{L}(\frac{2\Omega t}{\pi})^{\frac{1}{2}}] + 0(t) \end{aligned} \tag{246}$$

This demonstrates that even small viscosities result in a non- negligible net mass transport.

## 4.6 THE STOKES DRIFT

Assume that

$$\begin{aligned} \Omega &= 0 \qquad \text{non - rotating case} \\ V &= 0 \\ U &= W. \end{aligned}$$

In the non-rotating case, (238) becomes

$$\nu U_{zz} - U_t = \nu\eta_0^2\sigma k^3 e^{-2\beta t}[8e^{2kz} - \frac{4\gamma}{k}e^{\gamma z}(\cos\gamma z - \sin\gamma z)]. \tag{247}$$

A particular solution of (247) is given by

$$U^{(P)} = \eta_0^2\sigma k[e^{2kz} - \frac{2k}{\gamma}e^{\gamma z}(\sin\gamma z + \cos\gamma z)]e^{-2\beta t}. \tag{248}$$

A homogeneous solution of (247) can be obtained by considering the following boundary value problem

$$\nu U_{zz}^{(h)} - U_t^{(h)} = 0 \tag{249}$$

subject to the boundary conditions

$$U_z^{(h)} = -U_z^{(P)} = 2\eta_0^2\sigma k^3 e^{-2\beta t} \qquad \text{at } z = 0 \tag{250}$$

$$U^{(h)} \to 0 \qquad z \to -\infty \tag{251}$$

The initial condition is

$$U^{(h)} = 0 \qquad \text{at } t = 0. \tag{252}$$

Solution $U^{(h)}$ can be obtained by using the shifting and convolution properties of Laplace transforms

$$U^{(h)} = 2\eta_0^2\sigma k^2(\frac{\nu}{\pi})^{\frac{1}{2}}e^{-2\beta t}\int_0^t \frac{e^{2\beta A - \frac{z^2}{4\nu A}}}{A^{\frac{1}{2}}}dA. \tag{253}$$

Thus the complete solution is given by

$$U = U^{(h)} + U^{(P)}. \tag{254}$$

This solution is valid if there is no rotation in the fluid. It can be easily verified that when $\beta = 0$, $nu = 0$ and $(\frac{k}{\gamma}) \ll 1$, (253) reduces to

$$U^{(s)} = \eta_0^2\sigma k e^{2kz}. \tag{255}$$

This quantity is termed the Stokes Drift. The Stokes drift is a mass transport velocity (Lagrangian) which is the drift velocity of the individual particles. This drift is meant for the regular monochromatic waves. However, for a random wave field the Stokes drift becomes to second order (Huang 1979):

$$U^{(s)} = \int_0 \int_\sigma 2\sigma k e^{2kz} dX(k,\sigma) \tag{256}$$

where $X(k,\sigma)$ is the wave number-frequency spectrum of the variance of the sea surface elevation.

# 5 CONCLUDING REMARKS

Perrie and Rahman (1991) have computed the linear drift force components on isolated cylinders in regular arrays. In particular they have found that forces in the wave propagation direction and lateral to the wave propagation direction depend on the parameters, cylinder radius, separation distance and incident wavelength. These forces experience cancellation and enhancement (resonances) depending on values considered for parameters. A more fundamental calculation is considered by Rahman et al (1991). They present the second order drift forces acting on two cylinders standing upright in the water. As a function of separation distance, cylinder radius and incident wavelength, they compute the components of waterline and dynamic drift forces and show the circumstances in which resonances and mutual cancellation of these force components occur. Section 2 presents the basis for these drift force calculations and points beyond these calculations to an algorithm (Section 2.4) for estimation of the motion of floating bodies, such as ice floes. As yet we have not completed a model which would implement this algorithm.

Rahman and Perrie (1990) estimated the diffraction of a current field caused by a floe as discussed in Section 3. Their analysis used boundary element analysis to predict the flow pattern around the floe. They were able to consider geophysically reasonable boundary flows for upstream and downstream and obtain resultant streamlines as expected physically.

Finally, Perrie and Hu (1991) are implementing the theory of Weber (1983) and Jenkins (1991) as set forth in Section 4 for the surface current induced by forcing winds and waves. When an acceptable simulation of these currents is achieved, a modelling of ice floe trajectories will be attempted.

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# REFERENCES


1. Bai, K.J., "Zero-Frequency Hydrodynamics Coefficients of Vertical Axisymmetric Bodies at a Free Surface," *Journal of Hydronautics*, Vol. 11, 1977, pp. 53-57.

2. Brebbia, C.A. *The Boundary Element Method for Engineers*, Pentech Press, London, 1984.

3. Chandler, P.C., "A Standard Statistical Approach to Modelling Iceberg Drift," *Proceedings of the 5th International Offshore Mechanics and Arctic Engineering Symposium*, Tokyo, Japan, Vol. IV, 1986, pp. 426-431.

4. Connor, J.J. and Brebbia, C.A. *Finite Element Methods for Fluids*, 2nd edn, Newnes-Butterworth, 1977.

5. Ekman, V.W., "On the Influence of the Earth's Rotation on Ocean-Currents," *Ark. Mat. Astron. Fys.*, 2(11), pp. 1-53.

6. Garrison, C.J., "Hydrodynamic Loading of Large Offshore Structures. Three-Dimensional Source Distribution Methods," *Numerical Methods in Offshore Engineering*, eds. O.C. Zienkiewicz, R.W. Lewis, and K.G. Stagg, Wiley, Chichester, England, 1978, pp. 87-140.

7. Garrison, C.J. and Chow, P.Y., "Wave Forces on Submerged Bodies", *J. Waterways, Harbours and Coastal Eng. Div.*, ASCE, 98, WW3, 1972, pp. 375-392.

8. Gill, A.E., *Atmosphere-Ocean Dynamics*, Academic Press, New York, 1982.

9. Hsiung, C.C. and Abul-Azm, A.F., "Iceberg Drift Affected by Wave Action", *Ocean Engineering*, Vol. 9, No. 5, 1982, pp. 433-440.

10. Huang, N.E., "On Surface Drift Currents in the Ocean", *J. Fluid Mech.*, 91, 1979, pp. 191–208.

11. Ikeda, M., "A Coupled Ice-Ocean Model of a Wind-Driven Coastal Flow", *J. Geophys. Res.*, Vol. 90, No. C5, 1985, pp. 9119–9128.

12. Isaacson, M., "Ice Mass Motions Near an Offshore Structure", *Proceedings of the 5th International Offshore Mechanics and Arctic Engineering Symposium*, Tokyo, Japan, Vol. I, 1986, pp. 441–447.

13. Jenkins, A.D., "The Use of a Wave Prediction Model for Driving a Near-Surface Current Model." In Press in Deutsches Hydro. Zeits, 1990.

14. John, F., "On the Motion of Floating Bodies, II, *Comm. Pure and Applied Math.*, Vol. 3, 1950, pp. 45–100.

15. Kellogg, D.D., *Foundation of Potential Theory.* Frederick Unger, New York, 1953.

16. Lamb, H., *Hydrodynamics*, 6th edition, Dover Publications, New York, 1932.

17. Lighthill, J., "Waves and Hydrodynamic Loading," *Proceedings of the 2nd International Conference on the Behaviour of Off-Shore Structures*, BOSS '79, London, Vol. I, pp. 1-40.

18. Liu, A.K. and Mollo-Christensen, E., "Wave Propagation in a Solid Ice Pack", *J. Phys. Ocean.*, Vol. 18, 1988, pp. 1702–1712.

19. Longuet-Higgins, M.S., "A Nonlinear Mechanism for the Generation of Sea Waves", *Proc. Roy. Soc. London*, Vol. A311, 1969, pp. 371–389.

20. Madsen, O.S., "A Realistic Model of the Wind-Induced Ekman Boundary Layer", *J. Phys. Oceanogr.*, Vol. 7, 1977, pp. 248–255.

21. Margin, H.C., "Finite Element Analysis of Fluid Flows", *Proc. Conf. Matrix Methods in Struct. Mech.*, AFFDL TR, 68–150, Patterson Air Force Base, Ohio, USA, 1969.

22. Mountain, D.G., "On Predicting Iceberg Drift", *Cold Regions Science and Technology*, Vol. 1, 1980, pp. 273–282.

23. Perrie, W. and Hu, Y., "Ice Floe Trajectories on a Rotating Ocean in Response to Forcing by Wind and Waves", In preparation, 1991.

24. Perrie, W. and Rahman, M., "Scattering of Ocean Waves by Arrays of Cylinders With Rotational Symmetry." In press in Ocean Engng., 1990.

25. Pierson, W.J., "Perturbation Analysis of the Navier-Stokes Equations in Lagrangian Form With Selected Linear Solutions", *J. Geophys. Res.*, 67, 1962, pp. 3151–3160.

26. Rahman, M. and Perrie, W., "Boundary Element Analysis for the Flow Around a Submerged Ice Floe", *Hydrosoft J.*, Vol. 3, No. 2, 1990, pp. 69–72.

27. Rahman, M., Satish, M.G., Perrie, W. and Zhu, J., "Drift Forces on a Pair of Vertical Cylinders". Submitted to *J. Fluid Mech.*

28. Smith, D.S. and Banke, E.G., "The Influence of Winds, Currents and Towing Forces on the Drift of Icebergs," *Cold Regions Science and Technology*, Vol. 6, 1982, pp.

241-255.

29. Sodhi, D.S. and El-Tahan, M., "Prediction of an Iceberg Drift Trajectory During a Storm", *Annals of Glaciology*, Vol. 1, 1980, pp. 77-82.

30. Squire, V., "Numerical Simulation of Ice Floes in Waves," Scot Polar Research Institute, *Technical Report 81-1.*

31. Ünlüata, Ü. and Mei, C.C., "Mass Transport in Water Waves", *J. Geophys. Res.*, 75, 1970, pp. 7611–7618.

32. Wadhams, P., "The Seasonal Ice Zone," In: *Geophysics of Sea Ice*, N. Untersteiner [editor], Plenum Publishing Corporation, 1986, pp. 825-991.

33. Weber, J.E., "Attenuated Wave-Induced Drift in a Viscous Rotating Ocean", *J. Fluid Mech.*, 137, 1983, pp. 115–129.

34. Wehausen, J.V. and Laitone, E.V., "Surface Waves", in *Handbuch der Physik*, ed. S. Flugge, Springer-Verlag, Berlin, Vol. IX, 1960, pp. 446–778.